Strong (D)QBF Dependency Schemes via Tautology-free Resolution Paths

Olaf Beyersdorff Joshua Blinkhorn **Tomáš Peitl**

Friedrich-Schiller-Universität Jena, Germany

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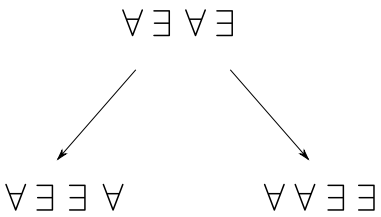
Dependencies

$$\forall \exists \forall \exists$$

Dependencies



Dependencies

DQBF

We consider (closed, prenex) *dependency quantified Boolean formulas* of the following form (a.k.a. *S-form DQBF*):

$$\Psi = \underbrace{\overbrace{\forall u_1}^{\text{universal variable}} \cdots \forall u_m \quad \overbrace{\exists x_1}^{\text{existential variable}}(S_{x_1}) \cdots \exists x_n \overbrace{(S_{x_n})}^{\text{support set}}}_{\text{prefix}} \cdot \underbrace{C_1 \wedge \cdots \wedge C_r}_{\text{matrix}}$$

$$\underbrace{u_1 \vee \overbrace{\neg x_2}^{\text{literal}} \vee \neg u_3 \vee x_4}_{\text{clause}}$$

A DQBF is *true* if there exist functions $f_{x_i} : \{0,1\}^{S_{x_i}} \to \{0,1\}$ whose substitution for x_i yields a propositional tautology.

QBF

- DQBF extends QBF:

$$\Phi = \overbrace{\forall U_1}^{\text{block}} \exists X_1 \forall U_2 \exists X_2 \cdots \forall U_k \exists X_k \cdot C_1 \wedge \cdots \wedge C_r$$

- If $x_i \in X_i$, then $S_{x_i} = \bigcup_{j<i} U_j$.
- A DQBF is a QBF if and only if the support sets are linearly ordered under inclusion.

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- Deciding whether a given DQBF is true is NEXP-complete.
- DQBFs can be used to model various real-world problems arising in areas such as formal verification, synthesis, automated design of circuits, or games such as chess.
- We are interested in solving DQBFs as efficiently as possible.

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- But that does not need the dependency on u.
- Hence, the dependency of x on u is spurious.

Dependency Schemes

- A *dependency scheme* as defined for QBF is a mapping:

$$D : \Phi \mapsto D(\Phi) \subseteq \mathcal{D}^{\text{trv}}(\Phi) = \{(x, y) \mid x < y\}$$

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- Because dependency schemes were created for QBF, dependencies are defined both ways. This turned out unnecessary in the analysis of refutational proof systems, and becomes meaningless in DQBF.

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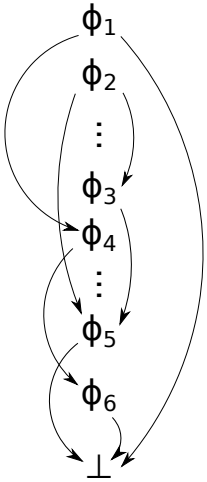

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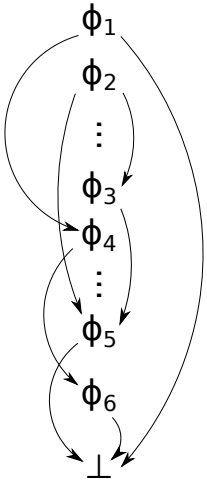

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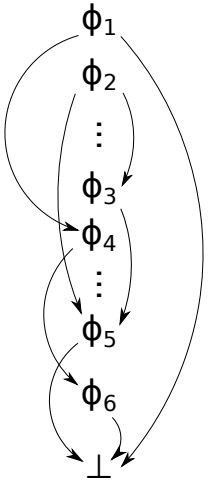

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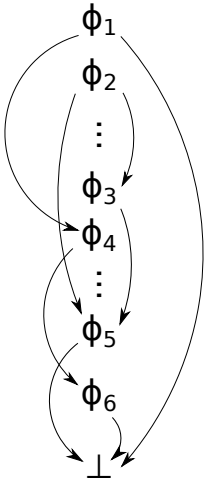

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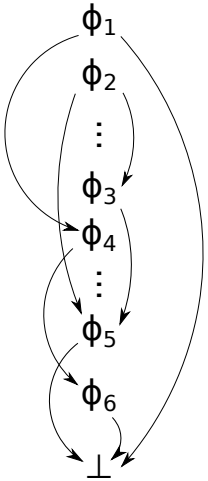
- A Q-Res refutation is a sequence of clauses that are either existential resolvents or universal reducts;
- A ∀Exp+Res refutation is a resolution refutation of the universally expanded formula (a.k.a. Shannon expansion);



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Theorem ([SS16])

A QBF is false if, and only if, it has a Q($\mathcal{D}^{\text{rrs}}$,$\mathcal{D}^{\text{std}}$)-Res refutation.

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- trying to solve a DQBF;
- identify as many spurious dependencies as possible;
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1. preserves the matrix and does not enlarge support sets:

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We say that a dependency scheme $\mathcal{D}$ is *fully exhibited* if $\mathcal{D}(\Psi) \overset{\text{tr}}{\equiv} \Psi$ for every DQBF Ψ.

Parameterising Proof Systems

Definition (P($\mathcal{D}$))

Let P be a DQBF proof system and let $\mathcal{D}$ be a dependency scheme. A P($\mathcal{D}$) refutation of a DQBF Ψ is a P refutation of $\mathcal{D}(\Psi)$.

Proposition

Given a DQBF proof system P *and a dependency scheme* $\mathcal{D}$, P($\mathcal{D}$) *is sound and complete if, and only if,* $\mathcal{D}$ *is fully exhibited.*

The Tautology-free Dependency Scheme

Definition ($\mathcal{D}^{\mathrm{rrs}}$ [SS16] and $\mathcal{D}^{\mathrm{tf}}$)

The *reflexive resolution path dependency scheme* ($\mathcal{D}^{\mathrm{rrs}}$) is defined as the mapping $\Psi \mapsto \Psi'$, where

$$\begin{aligned} \Psi &:= \forall u_1 \cdots \forall u_m \exists x_1(S_{x_1}) \cdots \exists x_n(S_{x_n}) \cdot \psi\,, \\ \Psi' &:= \forall u_1 \cdots \forall u_m \exists x_1(S'_{x_1}) \cdots \exists x_n(S'_{x_n}) \cdot \psi\,, \end{aligned}$$

and S'_{x_i} is the set of universal variables $u \in S_{x_i}$ for which there exists a sequence $C_1, \ldots, C_k$ of clauses in ψ and a sequence $p_1, \ldots, p_{k-1}$ of existential literals satisfying the following conditions:

(a) $u \in C_1$ and $\overline{u} \in C_k$;

(b) for some $j \in [k-1]$, $x_i = \mathrm{var}(p_j)$;

(c) for each $j \in [k-1]$, $p_j \in C_j$, $\overline{p}_j \in C_{j+1}$, and $u \in S_{\mathrm{var}(p_j)}$;

(d) for each $j \in [k-2]$, $\mathrm{var}(p_j) \neq \mathrm{var}(p_{j+1})$.

The *tautology-free dependency scheme* ($\mathcal{D}^{\mathrm{tf}}$) adds to $\mathcal{D}^{\mathrm{rrs}}$ the condition

(e) for each $j \in [k-1]$, $(C_j \cup C_{j+1})\restriction_{I_\exists(\Psi)}$ is non-tautological.

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- A resolution path is a sequence of clauses which can trigger unit propagation under a suitable assignment;
- If a resolution path connects u and x, then assigning u may affect the choices for x;
- However, certain resolution paths are blocked: they contain tautologies on variables that are "already assigned at the time" u is assigned, such as the *independent* existential variables $I_{\exists}(\Psi)$.

Example ($\mathcal{D}^{\text{rrs}}$ vs. $\mathcal{D}^{\text{tf}}$)

$$\forall u \exists x(\emptyset) \exists z(\{u\}) \cdot (x \vee u \vee z) \wedge (\neg x \vee \neg u \vee \neg z)$$

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$$\forall u \exists x(\emptyset) \exists z(\{u\}) \cdot (x \vee u \vee z) \wedge (\neg x \vee \neg u \vee \neg z)$$

- The two clauses $(x \vee u \vee z)$ and $(\neg x \vee \neg u \vee \neg z)$ constitute a resolution path that connects u and z. Indeed, if x is set to false, the first clause simplifies to the implication $\neg u \implies z$, and if x is set to true, the second clause simplifies to $u \implies \neg z$. The value of u may potentially force either value of z.

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- But x has to be set "before" z, because it does not depend on anything. Hence one of the implications is always killed. In other words, the union of the clauses, restricted to independent existential variables, is a tautology.
- $\mathcal{D}^{\text{tf}}$ detects the tautology and concludes that z is independent of u. Indeed $x \mapsto 0$ and $z \mapsto 1$ is a model that exhibits this.

Properties of $\mathcal{D}^{\mathrm{tf}}$

Proposition

$\mathcal{D}^{\mathrm{tf}}$ is a monotone dependency scheme, i.e. $\Psi \leq \Psi' \implies \mathcal{D}^{\mathrm{tf}}(\Psi) \leq \mathcal{D}^{\mathrm{tf}}(\Psi')$.

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Proof.

By reduction to full exhibition of $\mathcal{D}^{\mathsf{rrs}}$ established for DQBF by Wimmer et al. [WSWB16]. If Ψ is true, pick a satisfying assignment α to $I_\exists(\Psi)$, and restrict with it. Because $\Psi[\alpha]$ has no independent existential variables, $\mathcal{D}^{\mathsf{tf}}$ reduces to $\mathcal{D}^{\mathsf{rrs}}$ and the theorem follows by full exhibition of $\mathcal{D}^{\mathsf{rrs}}$. □

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Corollary

$\mathcal{D}^{\mathrm{tf}}$ can be plugged in into any proof system, in particular $\forall$Exp+Res.

Separations

Genuine DQBF Separations

- Extending the notion of genuine QBF hardness [Che17, BHP20], we define *genuine DQBF separations* as such separations, where the hardness is not witnessed by any *embedded QBF family*.

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Definition

Let P and Q be DQBF proof systems. We write $Q \not\leq_p^* P$ when there exists a DQBF family $\{\Psi_n\}_{n\in\mathbb{N}}$ such that:

(a) $\{\Psi_n\}_{n\in\mathbb{N}}$ has polynomial-size Q refutations;

(b) $\{\Psi_n\}_{n\in\mathbb{N}}$ requires superpolynomial-size P refutations;

(c) every QBF family $\{\Phi_n\}_{n\in\mathbb{N}}$ with $\Phi_n \leq \Psi_n$ has polynomial-size P refutations.

We write $P <_p^* Q$ when both $P \leq_p Q$ and $Q \not\leq_p^* P$ hold.

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Definition (EQ_n^0 (adapted from [BBH19]))

$\text{EQ}_n^0 := \Pi_n^{\text{EQ}} \cdot \psi_n^{\text{EQ}}$, where

$$\Pi_n^{\text{EQ}} := \forall u_1 \cdots \forall u_n \exists x_1(\emptyset) \cdots \exists x_n(\emptyset)\, \exists z_1(u_1) \cdots \exists z_n(u_n)\,,$$

$$\psi_n^{\text{EQ}} := (\overline{z_1} \vee \cdots \vee \overline{z_n}) \wedge \bigwedge_{i=1}^{n} \Big((\overline{x_i} \vee \overline{u_i} \vee z_i) \wedge (x_i \vee u_i \vee z_i) \Big)\,.$$

Human readably:

- there are x_i and z_i depending on u_i such that for all values of the u_i
- if $u_i = x_i$, then z_i, but not all z_i.

First Separation

Theorem

$\{\mathsf{EQ}^0_n\}_{n\in\mathbb{N}}$ *requires exponential-size* $\forall$Exp+Res *refutations.*

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For all n, the dependency sets of $\mathcal{D}^{\mathsf{rrs}}(\mathsf{EQ}_n^0)$ *are empty.*

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Theorem ([BB19])

$\{\mathsf{EQ}^0_n\}_{n\in\mathbb{N}}$ *has linear-size* $\forall$Exp+Res$(\mathcal{D}^{\mathrm{rrs}})$ *refutations.*

Second Separation

Definition (EQ^1_n (adapted from [BB17]))

For each natural number n,

$$\mathsf{EQ}^1_n := \Pi^{\mathsf{EQ}}_n \exists r(\emptyset)\, \exists s(\{u_1, \ldots, u_n\}) \cdot \Big(\psi^{\mathsf{EQ}}_n \otimes (r \vee s)\Big) \wedge \Big(\psi^{\mathsf{EQ}}_n \otimes (\overline{r} \vee \overline{s})\Big) \wedge (r \vee \overline{s}) \wedge (\overline{r} \vee s)\,.$$

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Theorem

Hence, $\{\mathrm{EQ}^1_n\}_{n\in\mathbb{N}}$ requires exponential-size $\forall\mathrm{Exp}{+}\mathrm{Res}(\mathcal{D}^{\mathrm{rrs}})$ refutations, but has linear-size $\forall\mathrm{Exp}{+}\mathrm{Res}(\mathcal{D}^{\mathrm{tf}})$ refutations.

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- We defined a novel dependency scheme $\mathcal{D}^{\mathsf{tf}}$ based on the intuition about how resolution paths do and do not transfer information between variables;
- We showed that $\mathcal{D}^{\mathsf{tf}}$ is fully exhibited;
- We showed that the use of $\mathcal{D}^{\mathsf{tf}}$ in both $\forall$Exp+Res and Q-Res results in exponentially shorter proofs compared to $\mathcal{D}^{\mathsf{rrs}}$.

Summary

- We proposed a clean framework for DQBF dependency schemes and their proof complexity centered around the notion of full exhibition;
- We defined a novel dependency scheme $\mathcal{D}^{\mathsf{tf}}$ based on the intuition about how resolution paths do and do not transfer information between variables;
- We showed that $\mathcal{D}^{\mathsf{tf}}$ is fully exhibited;
- We showed that the use of $\mathcal{D}^{\mathsf{tf}}$ in both $\forall$Exp+Res and Q-Res results in exponentially shorter proofs compared to $\mathcal{D}^{\mathsf{rrs}}$.
- A short remark on the Equality formulas: the QBF template is hard for both proof system, but our DQBF version is only hard for $\forall$Exp+Res and becomes easy in Q-Res. Why?

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