Sorting Parity Encodings by Reusing Variables

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Problem Statement

Suppose we have a permutation σ on $[n]$. The two xor constraints:

$$x_1 \oplus x_2 \oplus \cdots \oplus x_n = 0 \quad \wedge \quad x_{\sigma(1)} \oplus x_{\sigma(2)} \oplus \cdots \oplus x_{\sigma(n)} = 1$$

are a contradiction.

Tseitin variables t_i and s_i represent the reordered parity:

$$\begin{array}{ll} t_1 = x_1 \oplus x_2 & s_1 = x_{\sigma(1)} \oplus x_{\sigma(2)} \\ t_{i-1} = t_{i-2} \oplus x_i & s_{i-1} = s_{\sigma(i-2)} \oplus x_{\sigma(i)} \\ t_{n-3} \oplus x_{n-1} \oplus \bar{x}_n & s_{\sigma(n-3)} \oplus x_{\sigma(n-1)} \oplus x_{\sigma(n)} \end{array} \qquad \text{for } i = 3 \text{ to } n-2$$

We can represent $a = b \oplus c$ in four clauses.

$$\bar{a} \vee b \vee c \qquad a \vee \bar{b} \vee c \qquad a \vee b \vee \bar{c} \qquad \bar{a} \vee \bar{b} \vee \bar{c}$$

An Easy Special Case

- A special case when the permutation is the identity map:

$$x_1 \oplus x_2 \oplus \cdots \oplus x_n = 0 \quad \wedge \quad x_1 \oplus x_2 \oplus \cdots \oplus x_n = 1.$$

- These formulas are instances of the `Dubois` family and have short linear resolution proofs.
- However when the reordered parity is done randomly, we do not know if there are short proofs for resolution.

SAT Solvers and Gaussian Elimination

- Appear to show exponential running times on CDCL algorithm.
- However there is at least one method that deals with reordered parity quickly: Gaussian elimination.
- Gaussian elimination requires the detection of xor constraints and then adds them together to create more xor constraints until an inconsistent constraint is found.
- However solvers that support Gaussian elimination cannot produce proofs.

The DRAT proof system

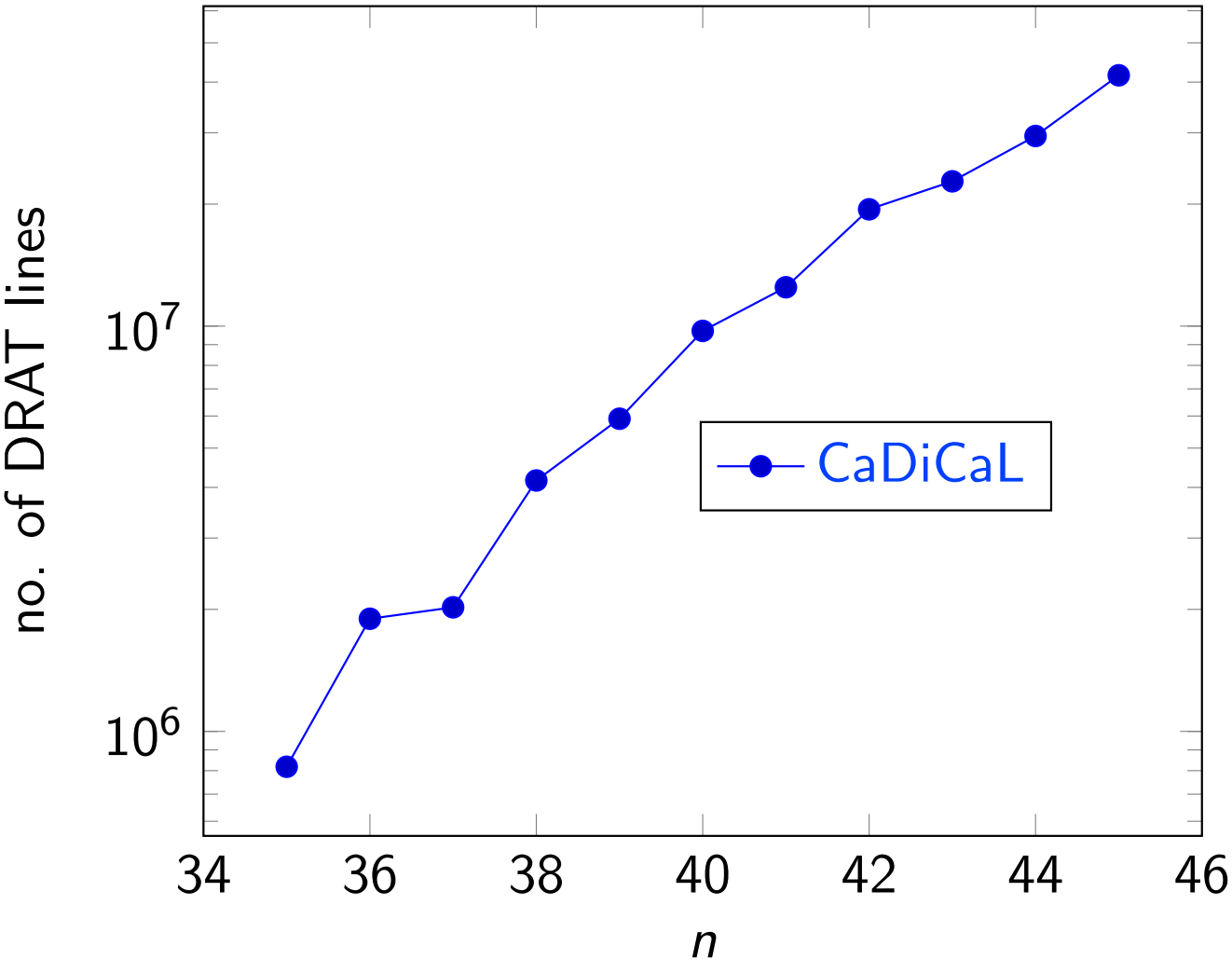
- Powerful proof system, simulates strong systems like Frege, Extended resolution (ER) and Propagation Redundancy (PR).
- Standard proof format in SAT solving.
- Every ER proof is automatically a DRAT proof.
- DRAT$^-$ is without new variables [Buss, Thapen 2019]

DRAT Rules:

- Logical implication rules (ATA, ATE)
- Satisfiability preserving rules (RATA, RATE)

Generated Proofs

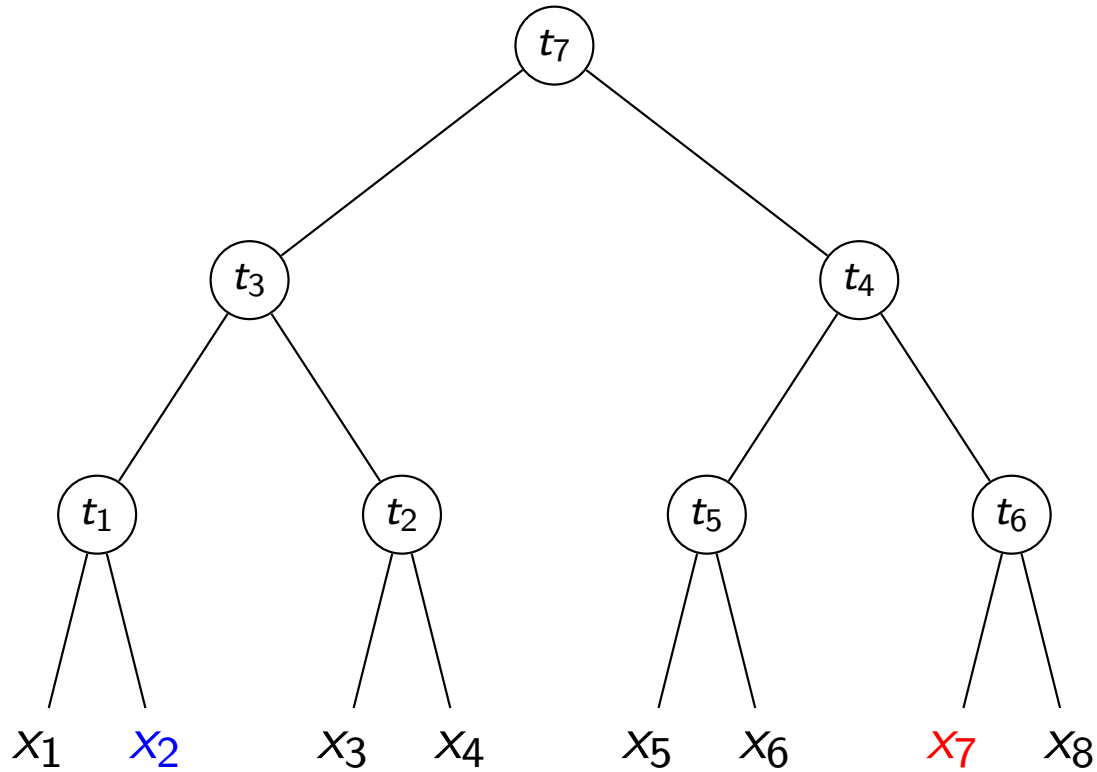
CDCL on reordered parity formulas appear exponential.



Solving Unordered Parity by Sorting

- We can use swaps to turn $x_{\sigma(1)} \oplus x_{\sigma(2)} \oplus \cdots \oplus x_{\sigma(n)} = 1$ into $x_1 \oplus x_2 \oplus \cdots \oplus x_n = 1$ (in Tseitinised form).
- Swaps must have short DRAT proofs.
- We limit swaps to adjacent variables, those used in the next Tseitin variable.
- The Tseitin variables are in a linear depth tree: hence $O(n^2)$ in the worst case.
- For a log depth tree this is $O(n \log n)$.

Sorting a Log Depth Tree



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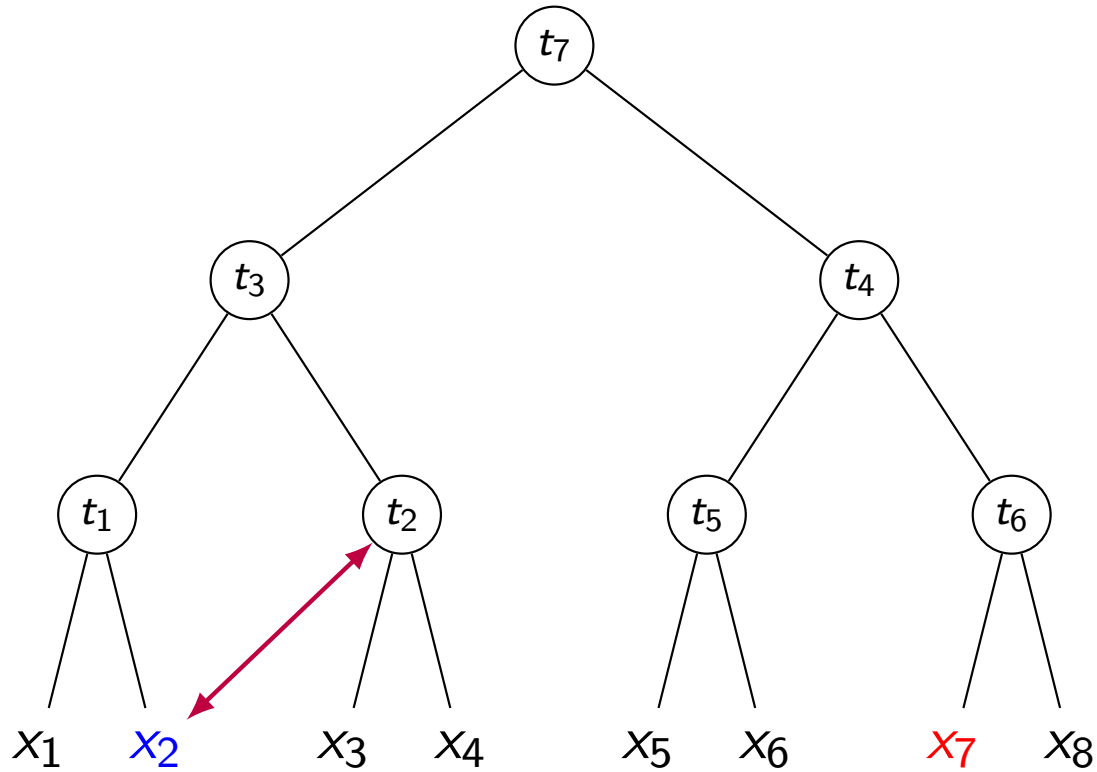

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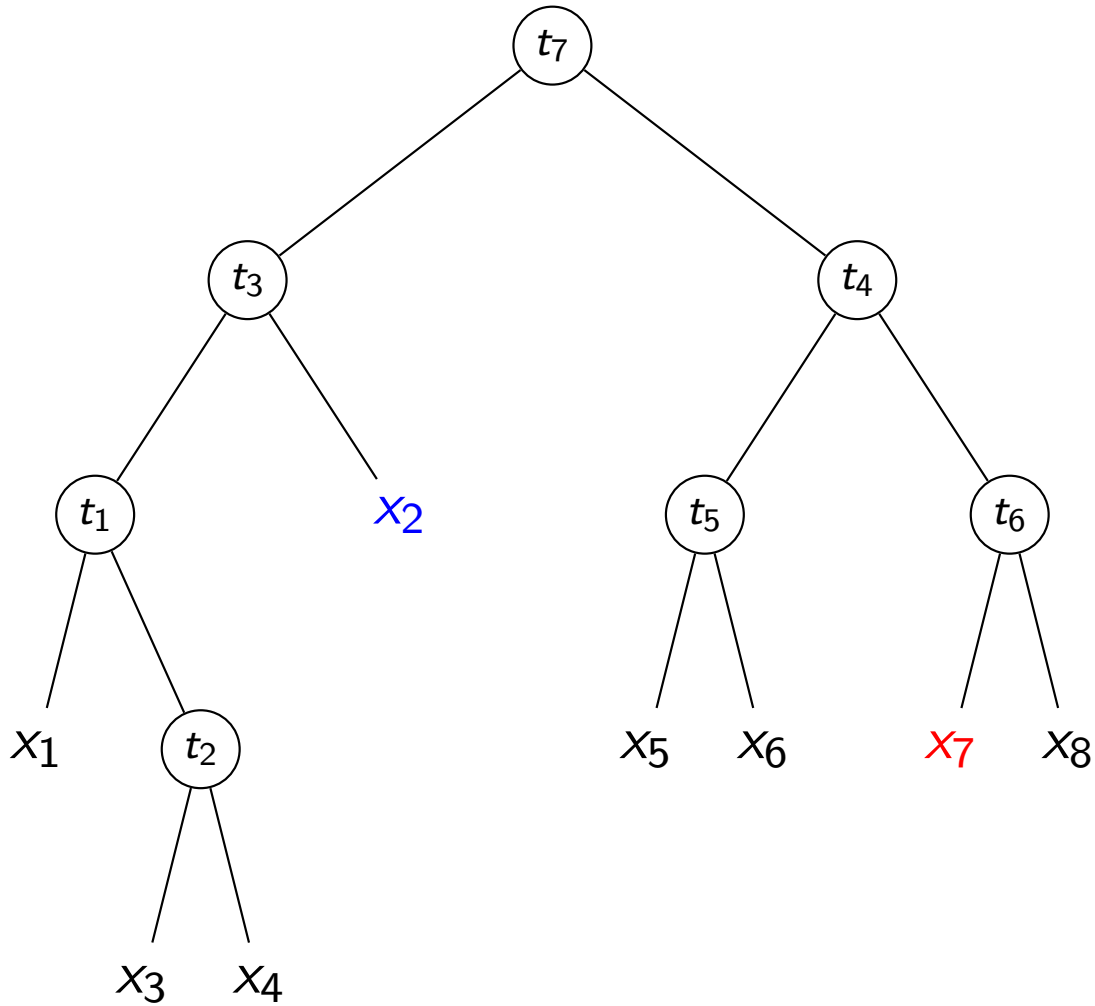

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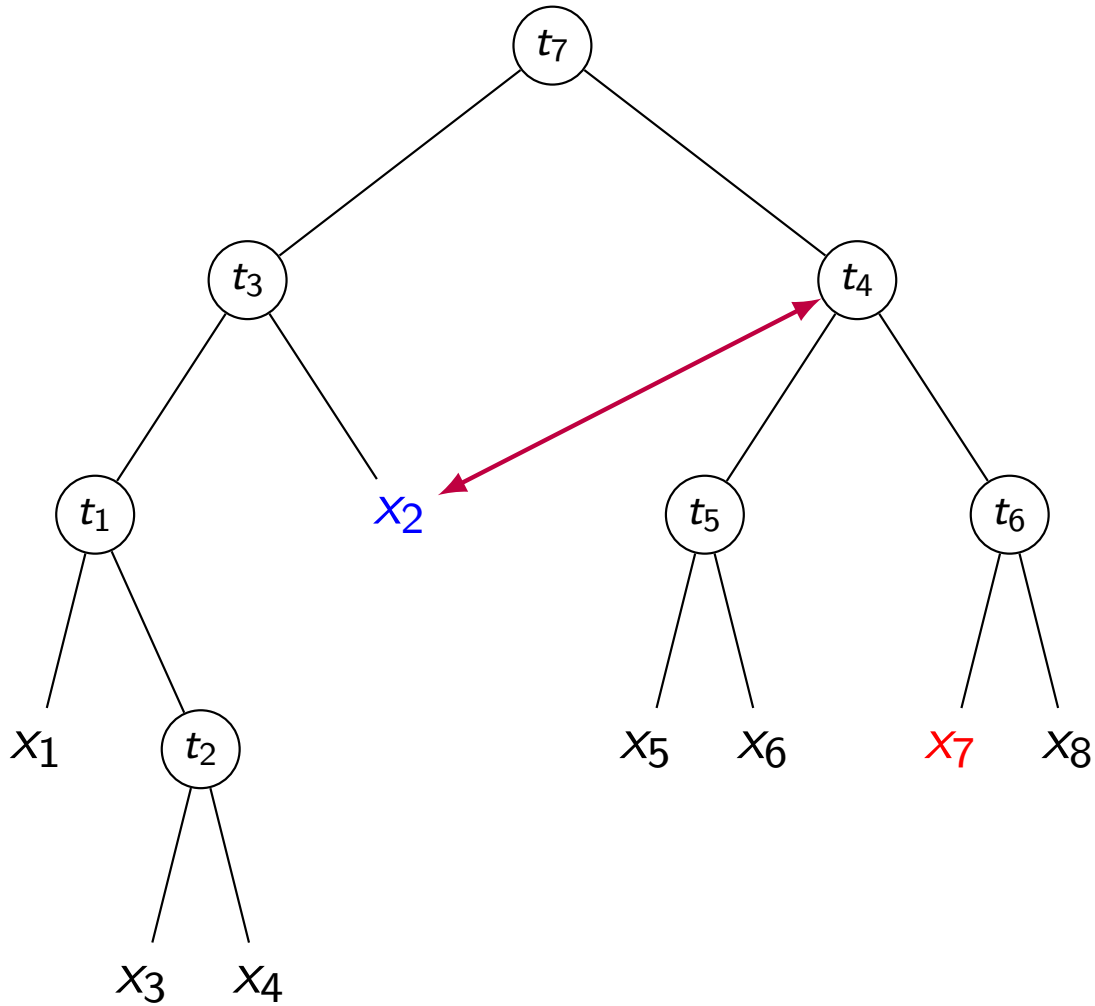

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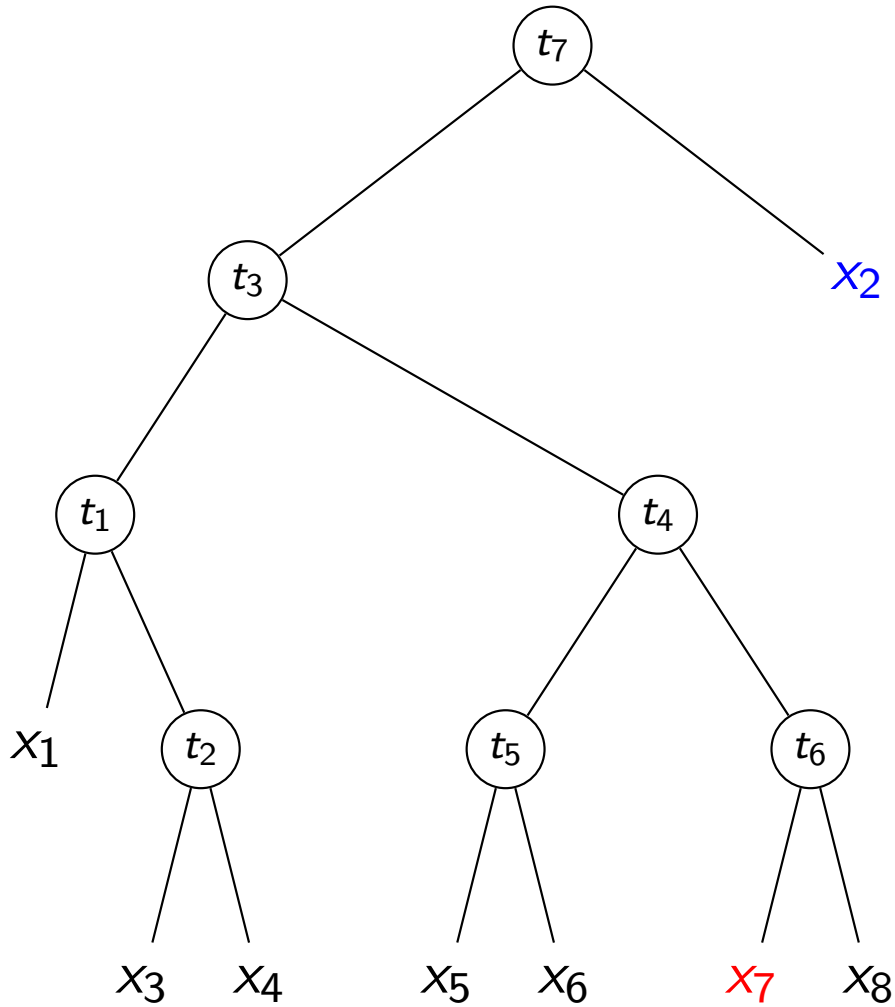

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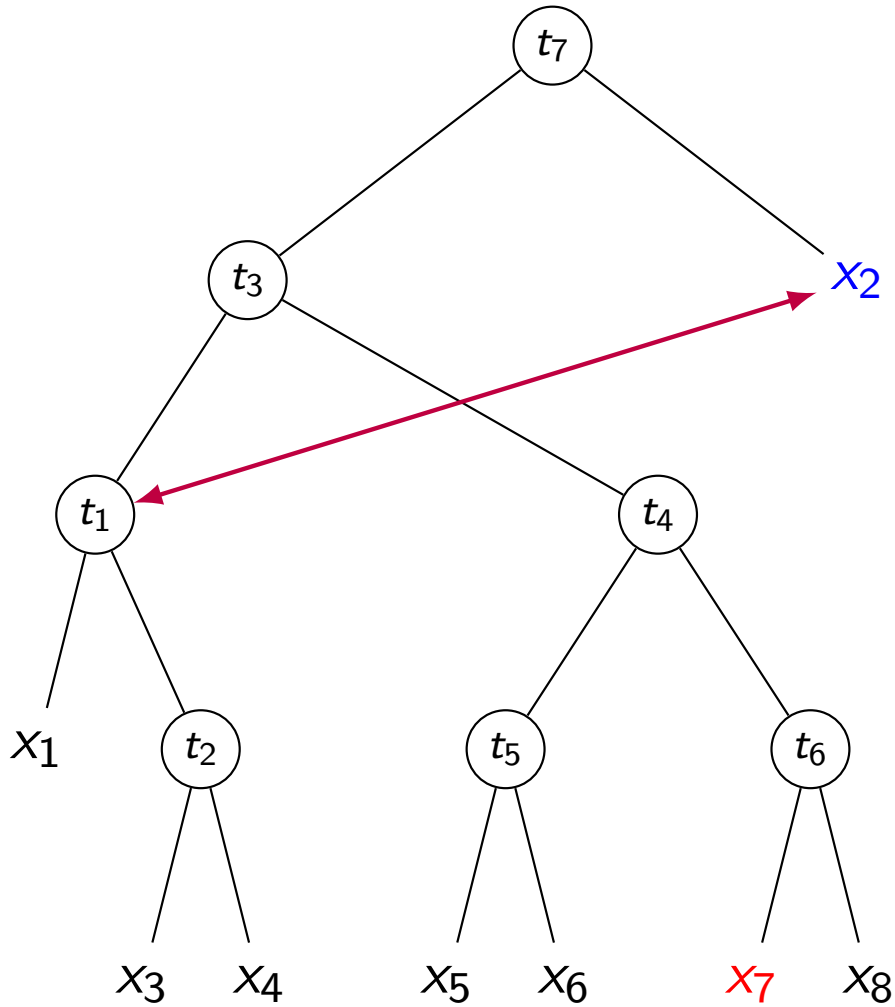

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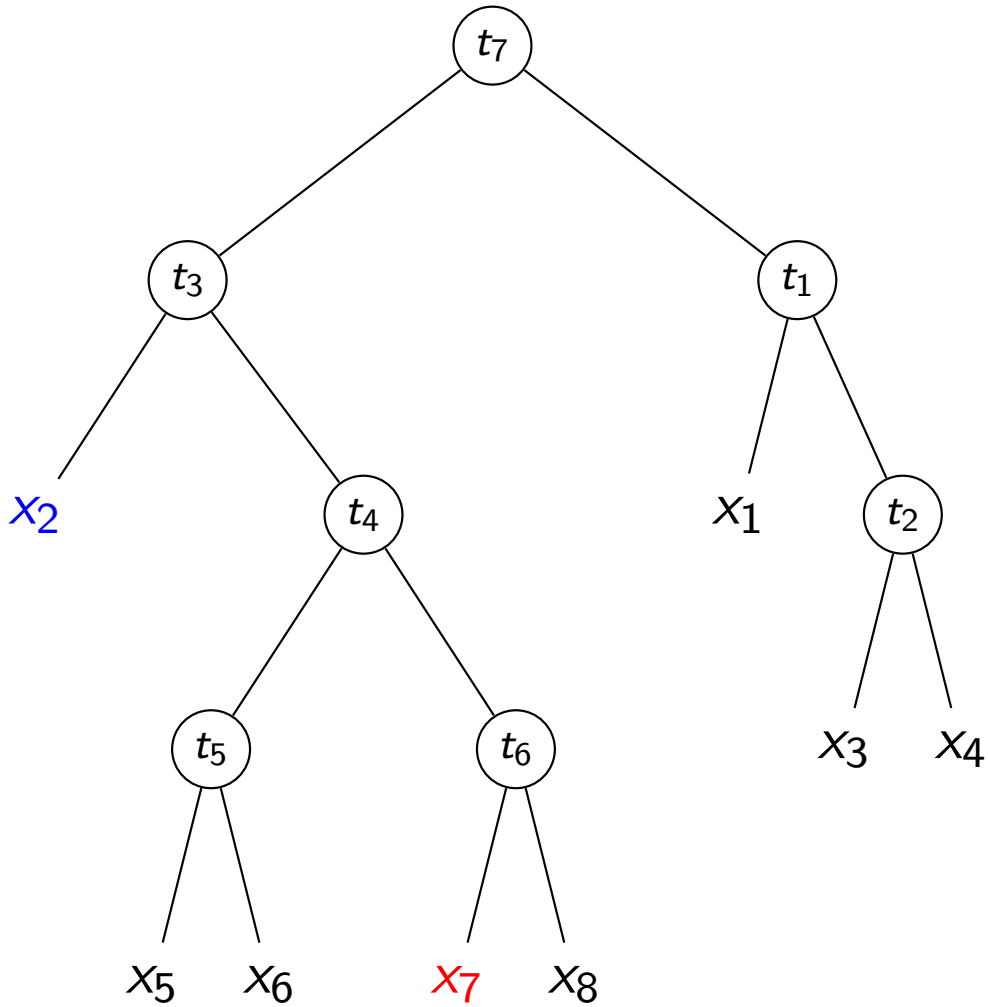

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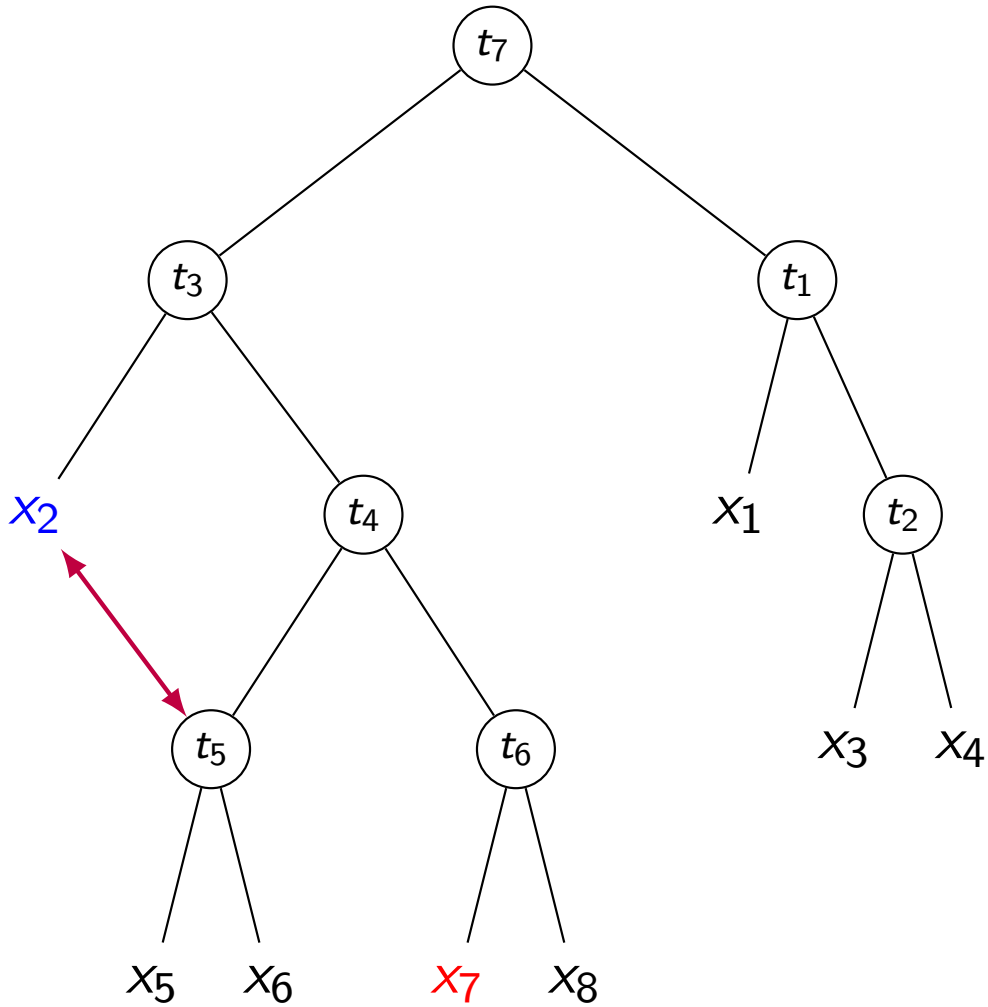

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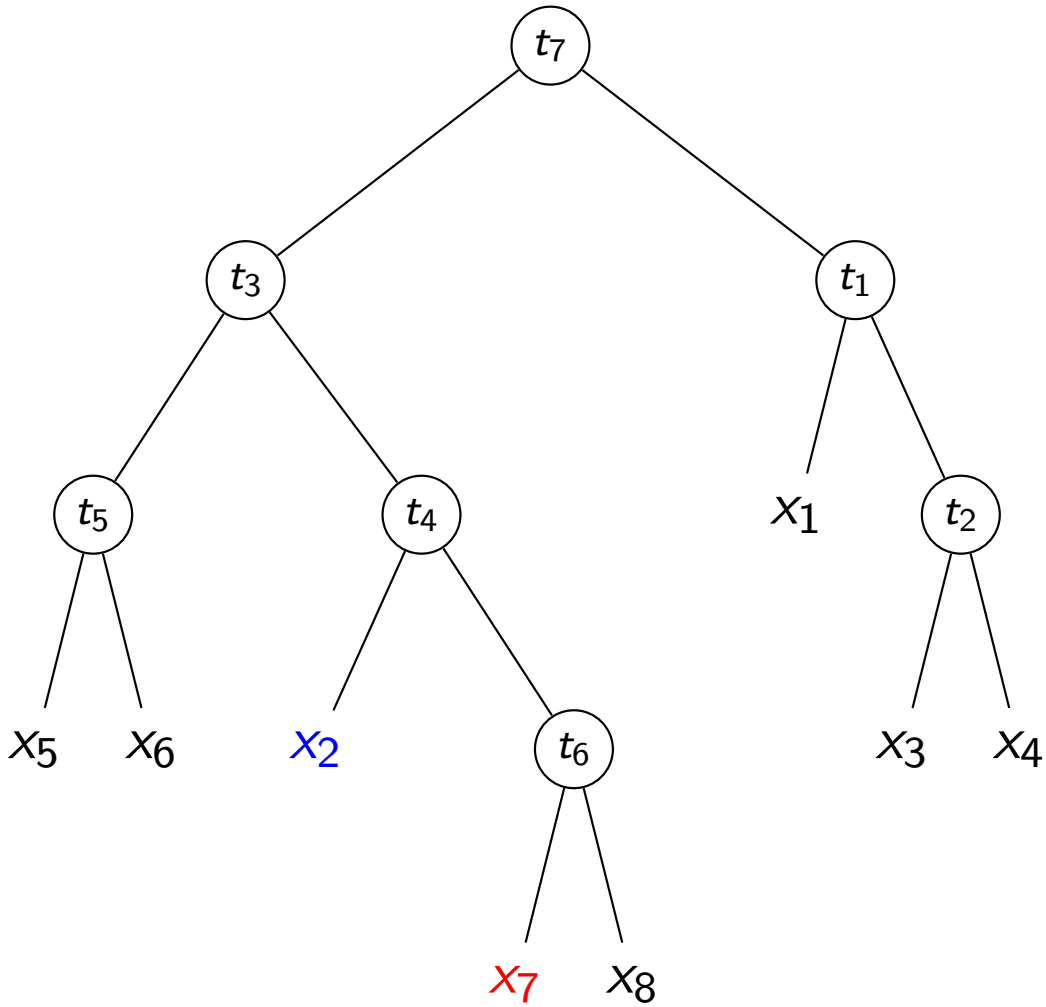

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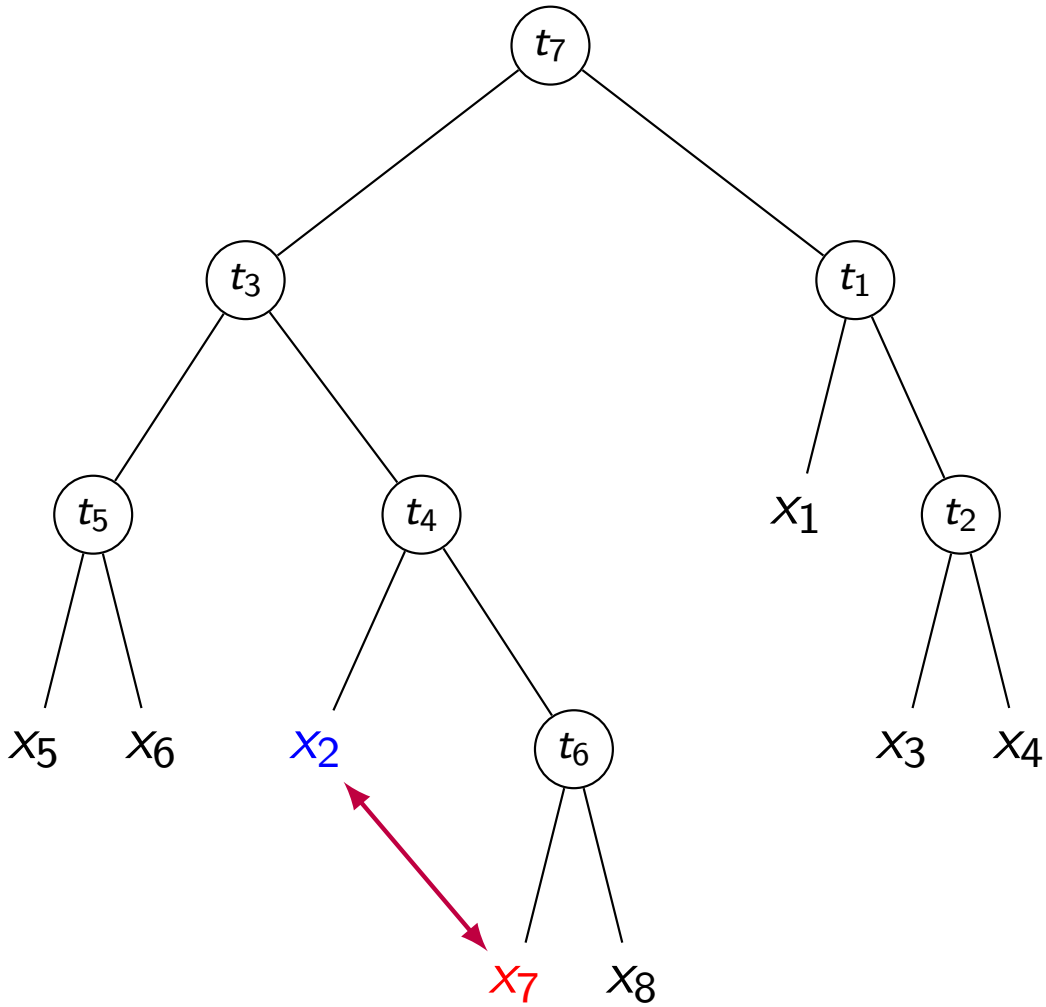

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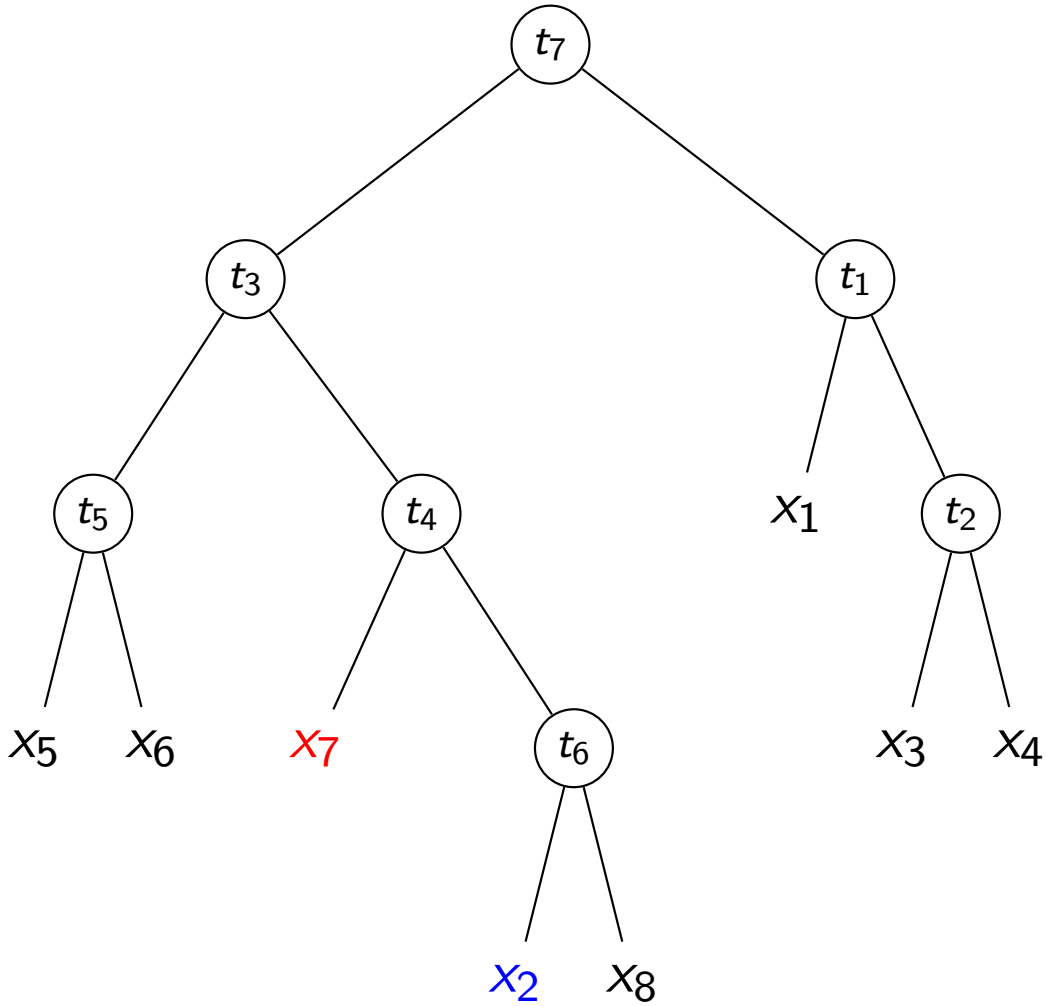

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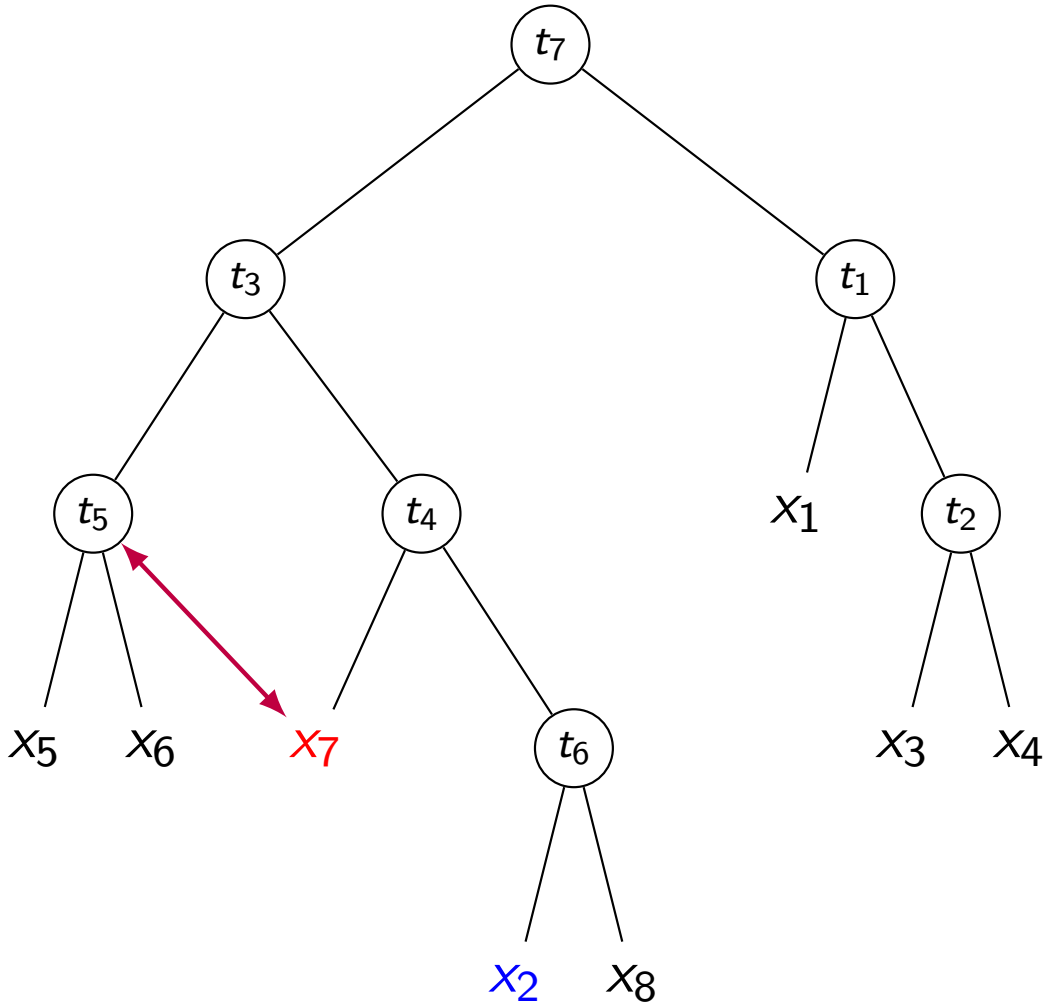

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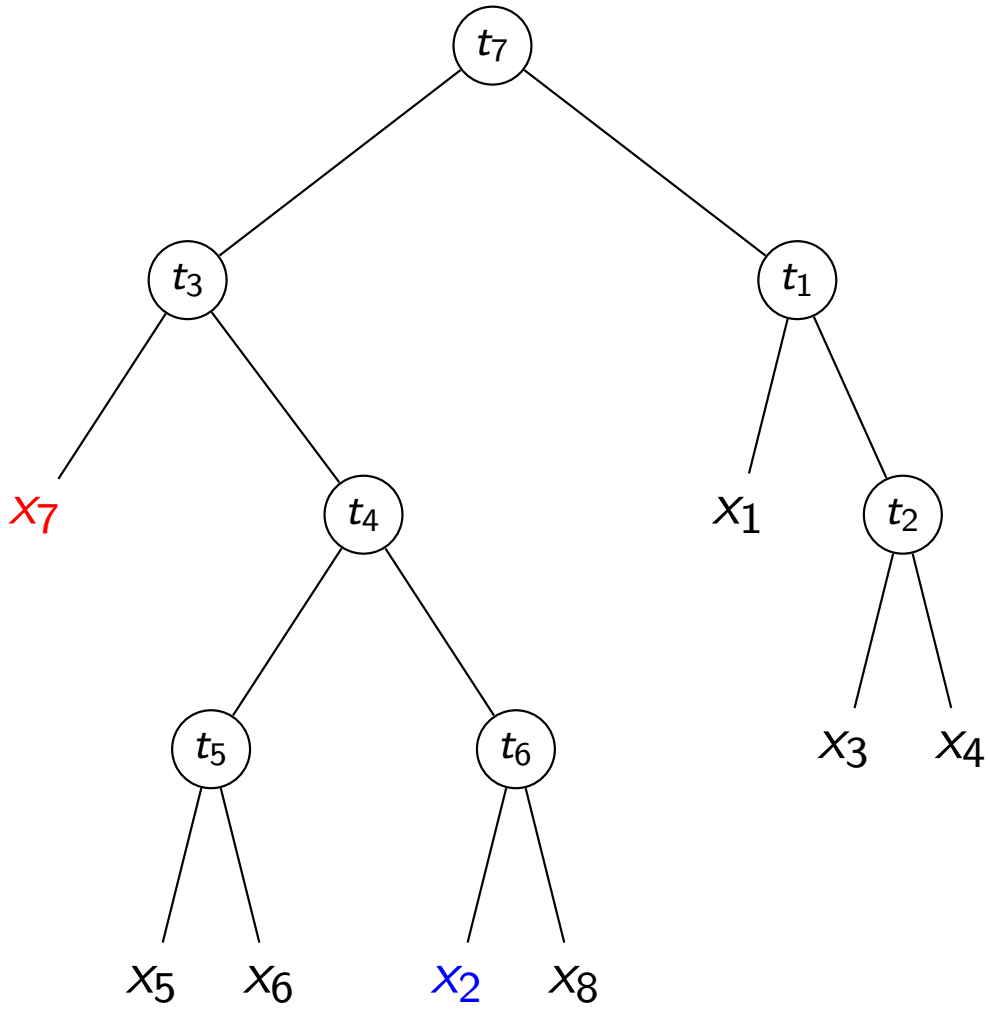

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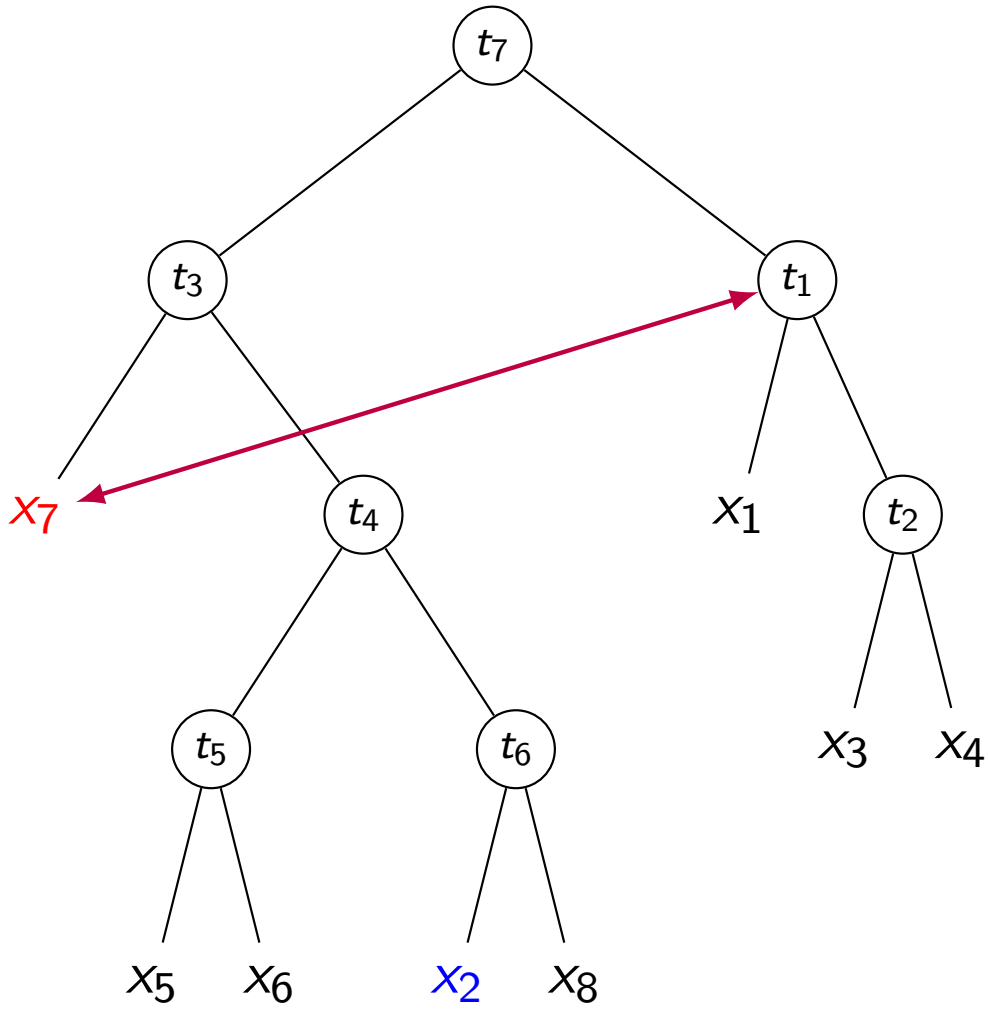

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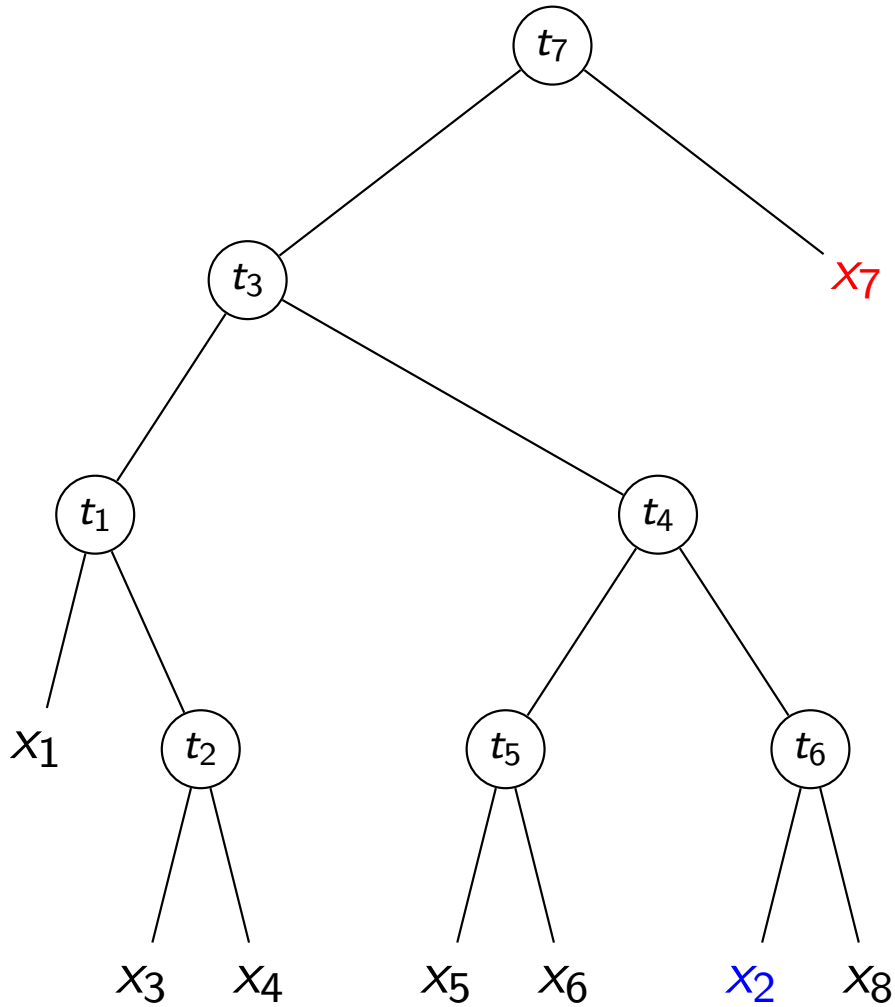

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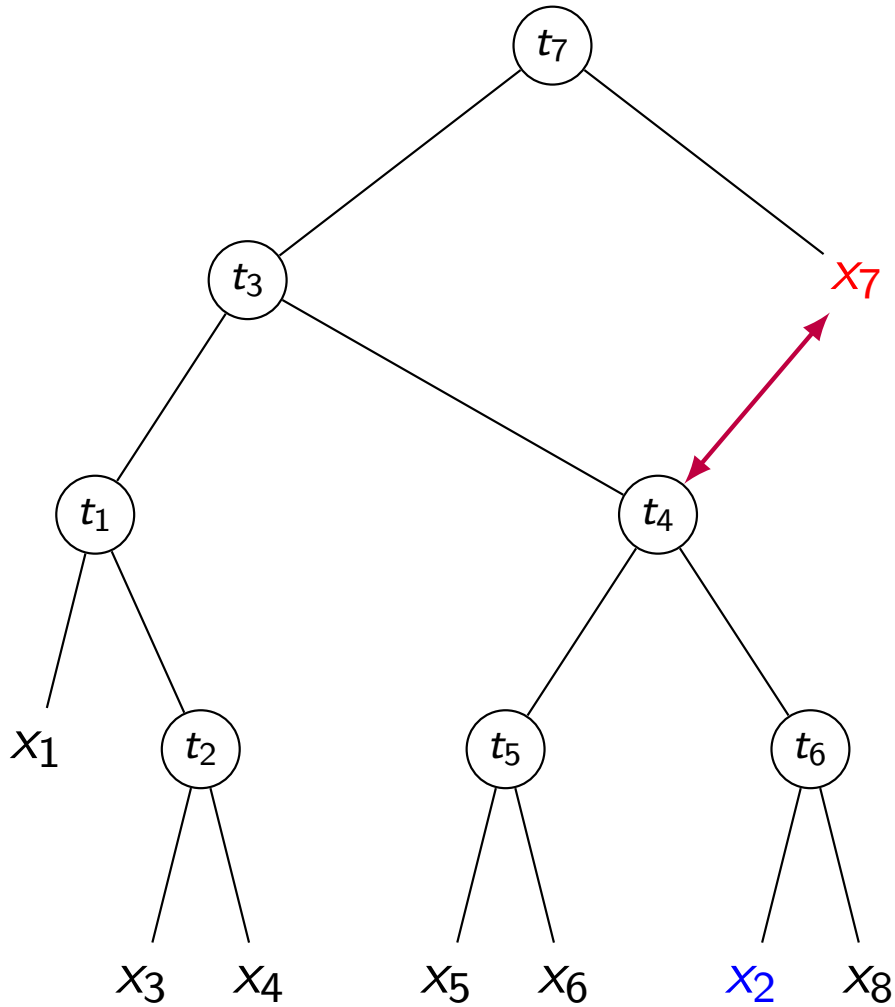

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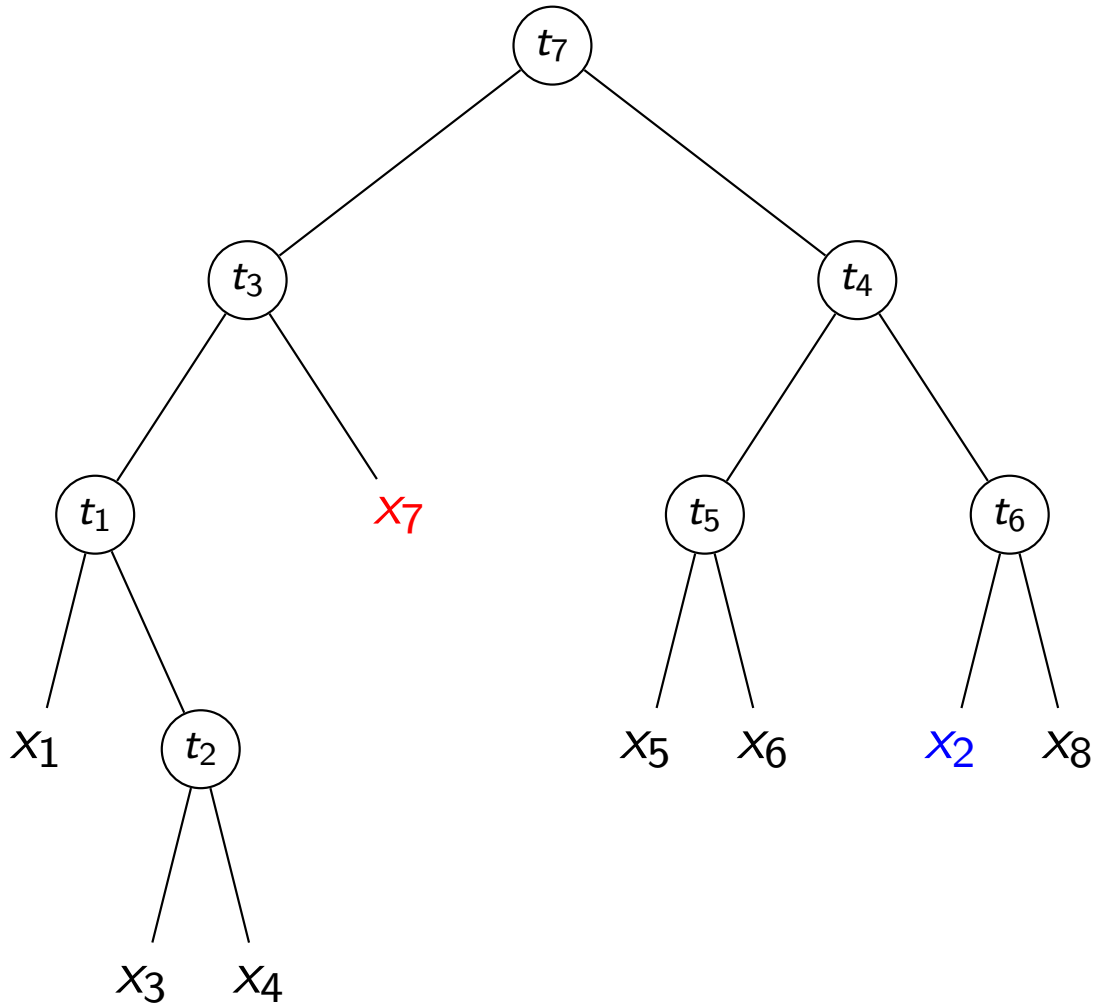

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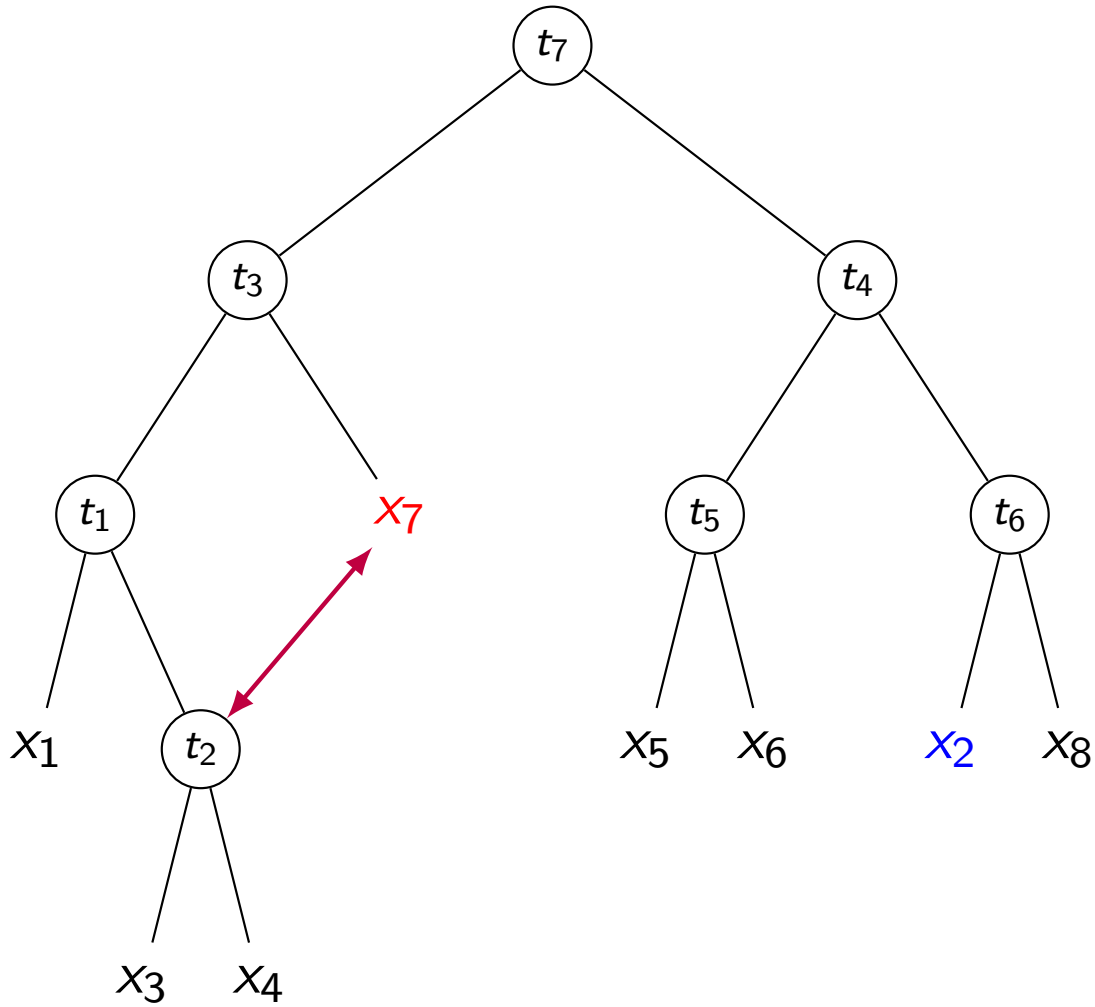

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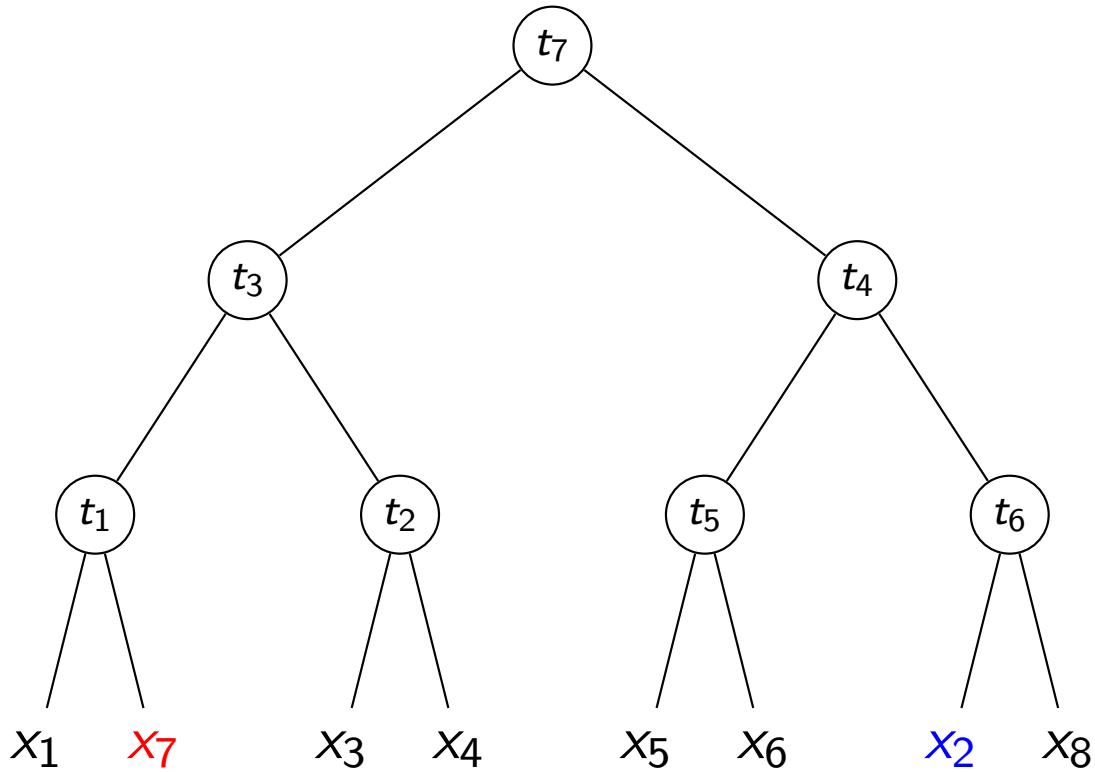
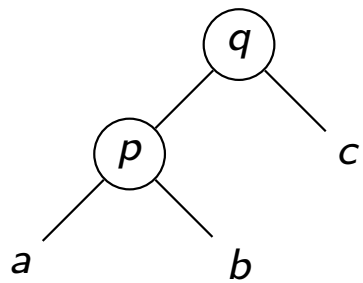


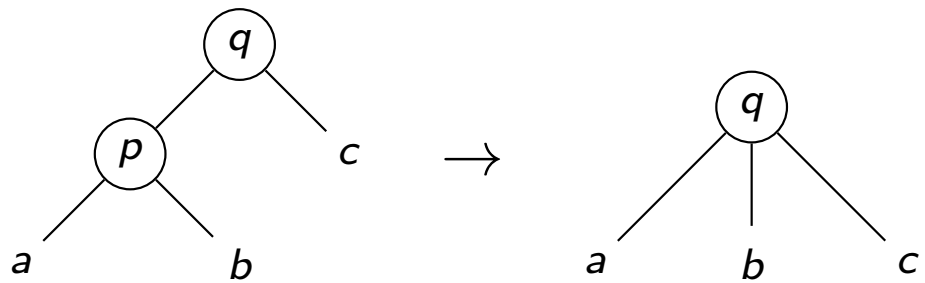
Logging Swaps in DRAT

Suppose we have two Tseitin variables p and q,
$p = a \oplus b$ and $q = p \oplus c$



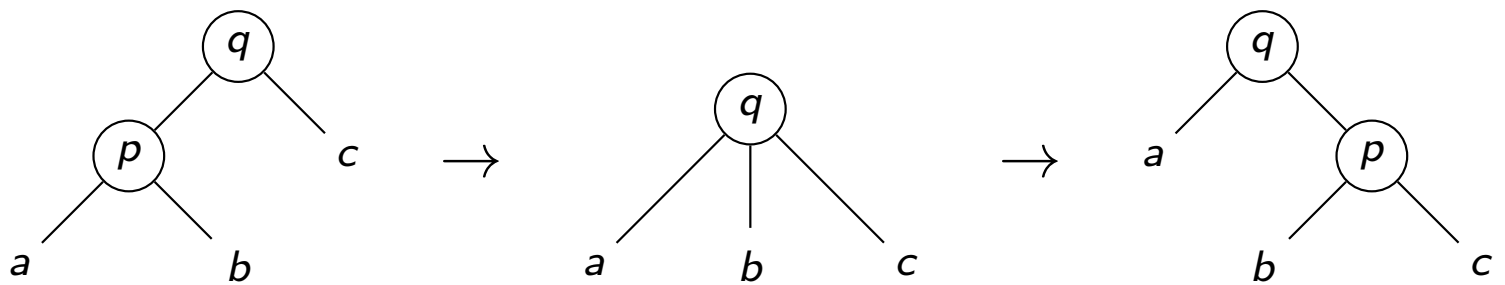
Logging Swaps in DRAT

ATA	RATE
$\bar{q} \vee a \vee b \vee c$	d $\bar{p} \vee a \vee b$
$\bar{q} \vee \bar{a} \vee \bar{b} \vee c$	d $\bar{p} \vee \bar{a} \vee \bar{b}$
$\bar{q} \vee a \vee \bar{b} \vee \bar{c}$	d $p \vee a \vee \bar{b}$
$\bar{q} \vee \bar{a} \vee b \vee \bar{c}$	d $p \vee \bar{a} \vee b$
$q \vee \bar{a} \vee b \vee c$	d $\bar{p} \vee q \vee c$
$q \vee a \vee \bar{b} \vee c$	d $\bar{p} \vee \bar{q} \vee \bar{c}$
$q \vee a \vee b \vee \bar{c}$	d $p \vee q \vee \bar{c}$
$q \vee \bar{a} \vee \bar{b} \vee \bar{c}$	d $p \vee \bar{q} \vee c$

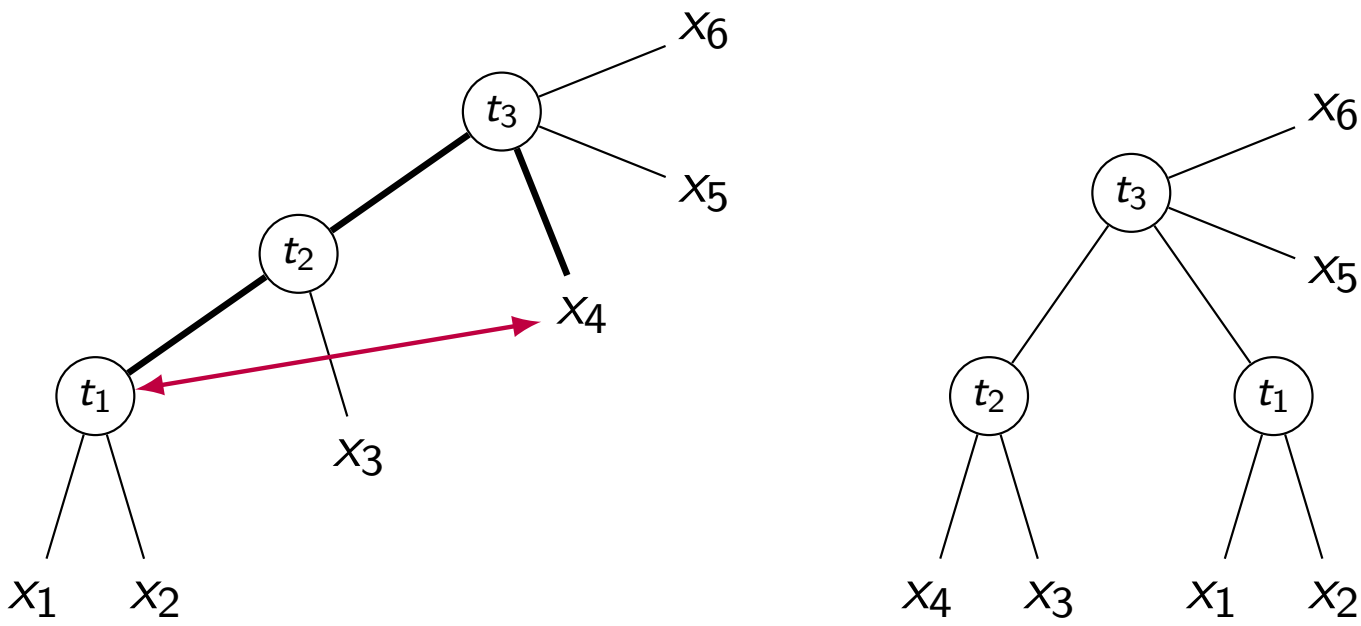


Logging Swaps in DRAT

ATA	RATE	RATA	ATE
$\bar{q} \vee a \vee b \vee c$	d $\bar{p} \vee a \vee b$	$\bar{p} \vee c \vee b$	d $\bar{q} \vee a \vee b \vee c$
$\bar{q} \vee \bar{a} \vee \bar{b} \vee c$	d $\bar{p} \vee \bar{a} \vee \bar{b}$	$\bar{p} \vee \bar{c} \vee \bar{b}$	d $\bar{q} \vee \bar{a} \vee \bar{b} \vee c$
$\bar{q} \vee a \vee \bar{b} \vee \bar{c}$	d $p \vee a \vee \bar{b}$	$p \vee c \vee \bar{b}$	d $\bar{q} \vee a \vee \bar{b} \vee \bar{c}$
$\bar{q} \vee \bar{a} \vee b \vee \bar{c}$	d $p \vee \bar{a} \vee b$	$p \vee \bar{c} \vee b$	d $\bar{q} \vee \bar{a} \vee b \vee \bar{c}$
$q \vee \bar{a} \vee b \vee c$	d $\bar{p} \vee q \vee c$	$\bar{p} \vee q \vee a$	d $q \vee \bar{a} \vee b \vee c$
$q \vee a \vee \bar{b} \vee c$	d $\bar{p} \vee \bar{q} \vee \bar{c}$	$\bar{p} \vee \bar{q} \vee \bar{a}$	d $q \vee a \vee \bar{b} \vee c$
$q \vee a \vee b \vee \bar{c}$	d $p \vee q \vee \bar{c}$	$p \vee q \vee \bar{a}$	d $q \vee a \vee b \vee \bar{c}$
$q \vee \bar{a} \vee \bar{b} \vee \bar{c}$	d $p \vee \bar{q} \vee c$	$p \vee \bar{q} \vee a$	d $q \vee \bar{a} \vee \bar{b} \vee \bar{c}$



Rebalancing



- Internal nodes can be swapped to balance the tree.
- By divide and conquer this is $O(n \log n)$ to make a linear tree log depth.

Complexity

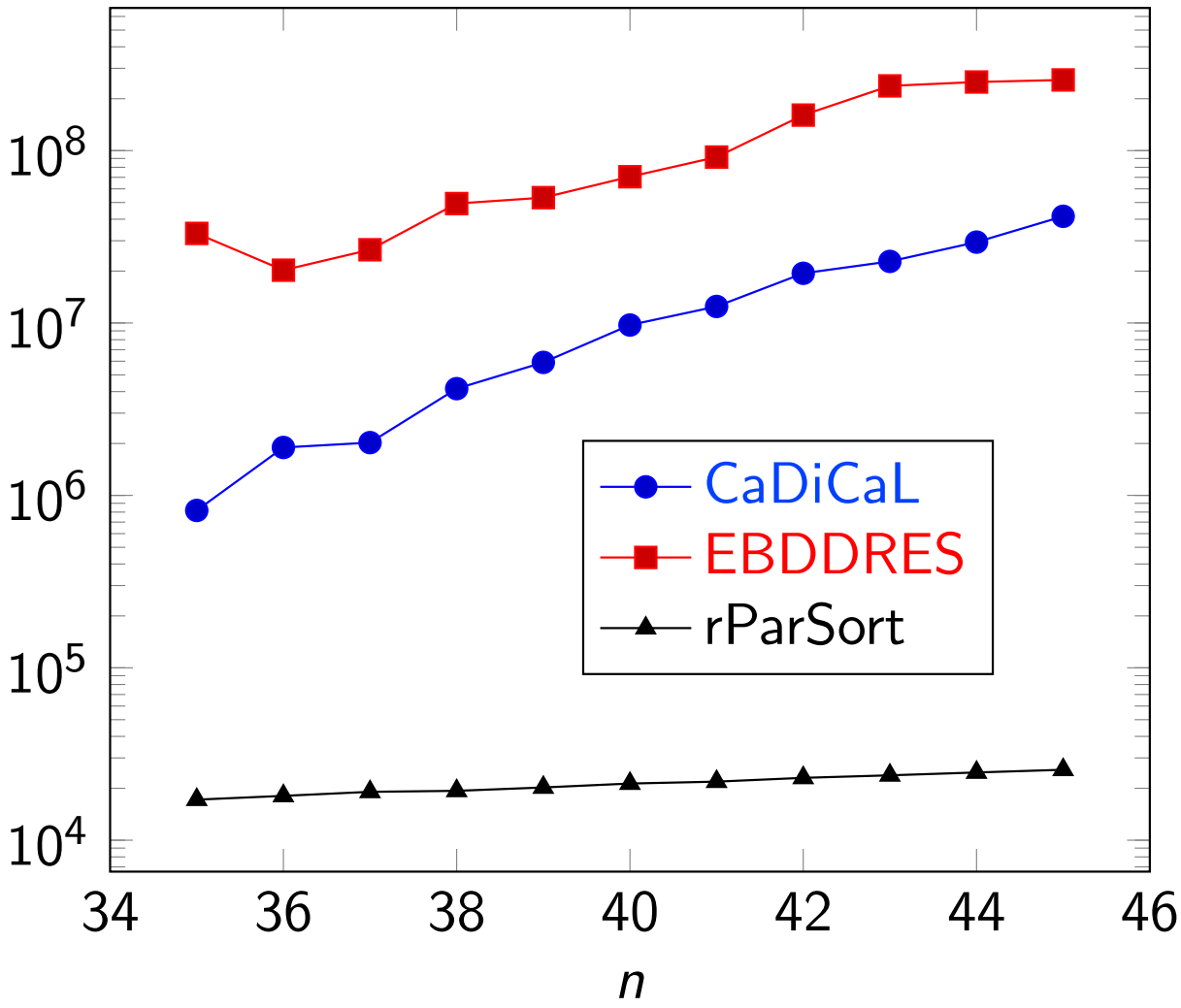
- Rebalancing the tree is $O(n \log n)$ because each level needs up to $n/2$ swaps. And there will be $\log n$ levels.
- Every transposition swap between two leaves, takes at most $4 \log n$.
- There is a maximum of n transpositions.
- The rebalancing needs to be undone which takes $O(n \log n)$
- The formulas are now in the `Dubois` family and can be refuted by an $O(n)$ size resolution proof.

$$O(n \log n) + n \cdot 4 \log n + O(n \log n) + O(n) = O(n \log n)$$

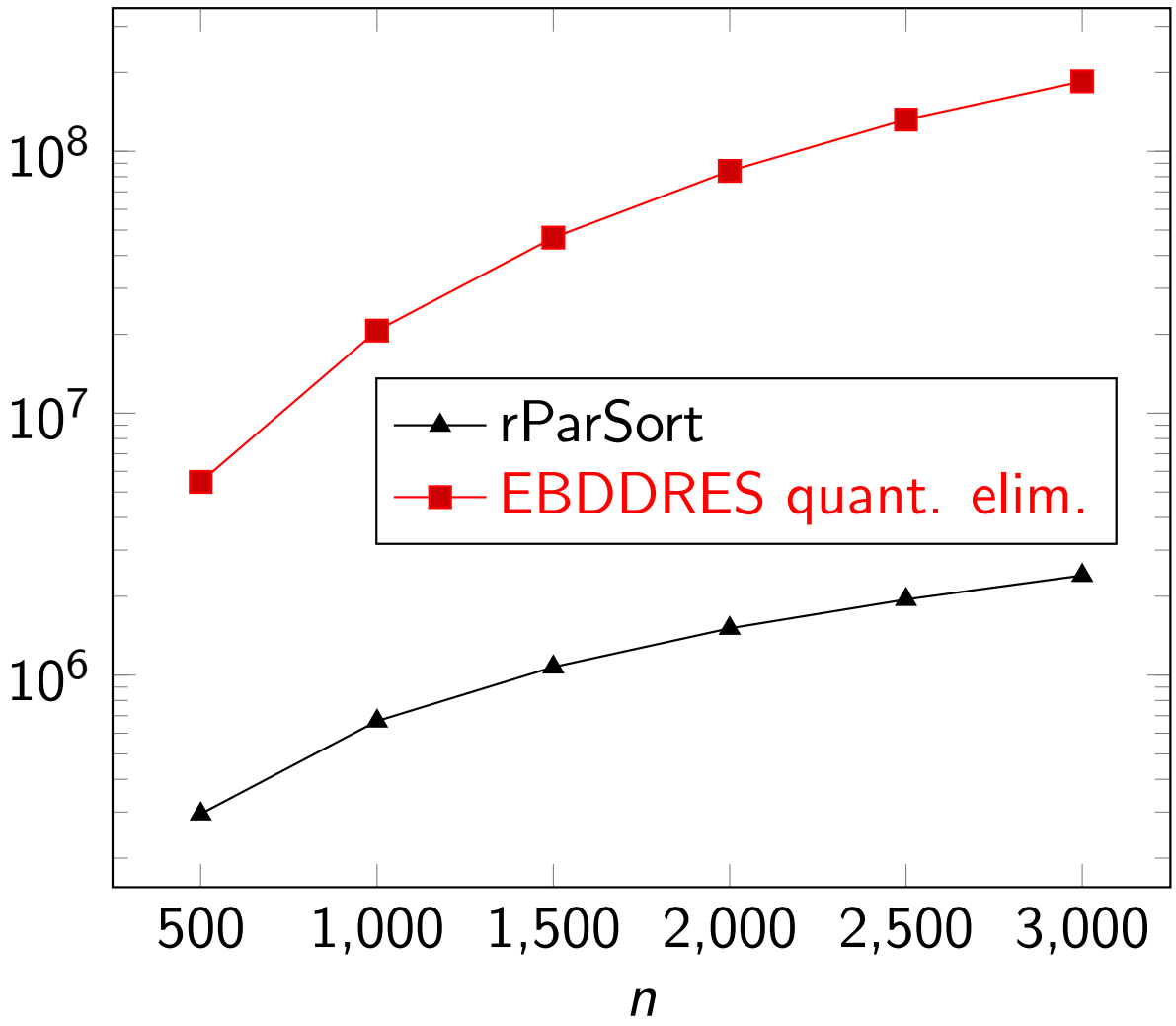
Experimental Setup

- We ran a program rParSort that generated a random instance of reordered parity and also generated a DRAT proof based on the aforementioned $O(n \log n)$ technique.
- rParSort proofs were very compact- 150MB for $n = 1000$.
- We compare the size of our proofs by ones produced by the state-of-the-art SAT solver CaDiCaL
- Another technique EBDDRES, solves the instance using binary decision diagrams and turns the construction into an ER proof.
- CaDiCaL and EBDDRES could manage up to $n = 45$, but started reaching > 5000 seconds time-outs, afterwards.

Comparisons I



Comparisons II (Not in Paper)



Summary

- Reordered parity constraints are hard for CDCL without Gaussian elimination
- Sorting the leaves can give an $O(n \log n)$ short proofs.
- The DRAT proof we have create no new variables, instead re-use existing ones.
- We have a working algorithm generating these short proofs, and these are outperforming all other existing tools.

Further Work

- The current algorithm assumes the structure of the tree, this can be made automatic.
- Reordered parity formulas are special cases of Tseitin graphs [Tseitin 1968]. We plan to generalise our technique for refutations of Tseitin formulas.
- The original problem was because Gaussian elimination does not support proofs. With adaptation we can use this sorting tool to help resolve two xor-constraints and simulate Gaussian elimination.