A Faster Algorithm for Propositional Model Counting Parameterized by Incidence Treewidth

Friedrich Slivovsky and Stefan Szeider

Propositional Model Counting (#SAT)

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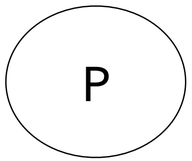

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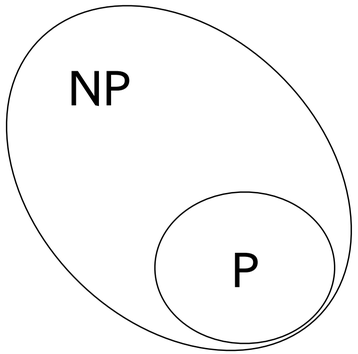

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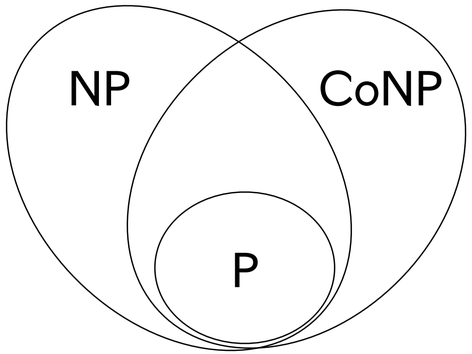

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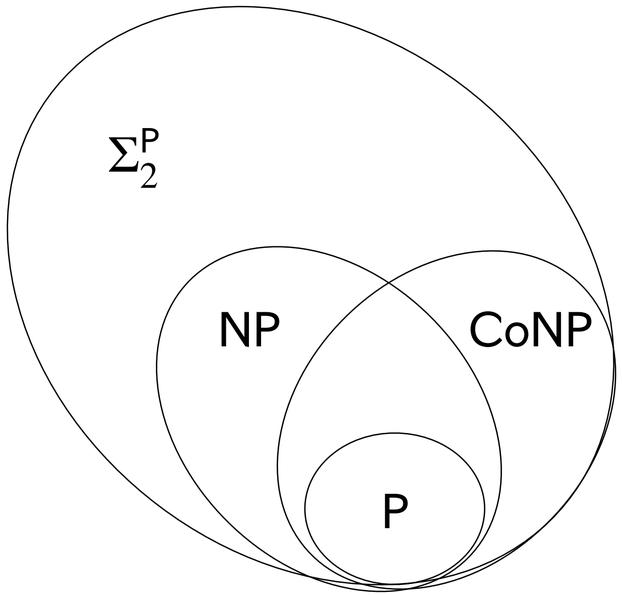

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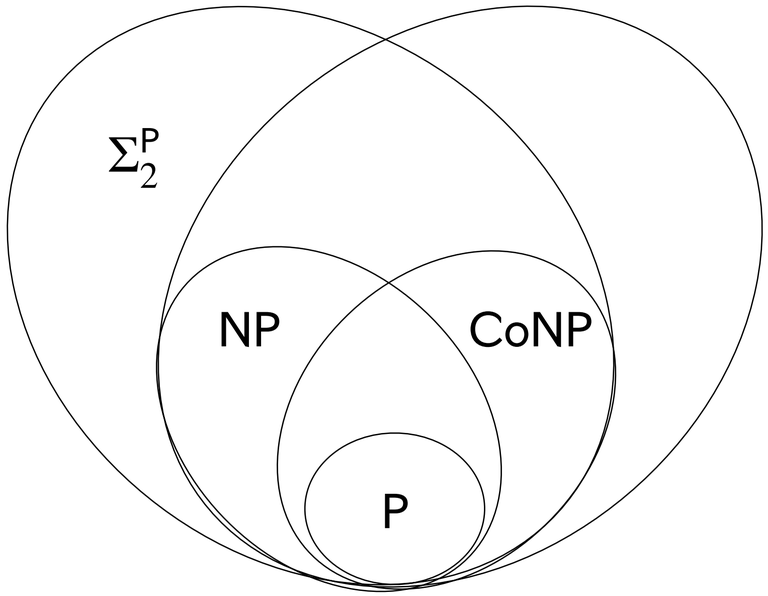

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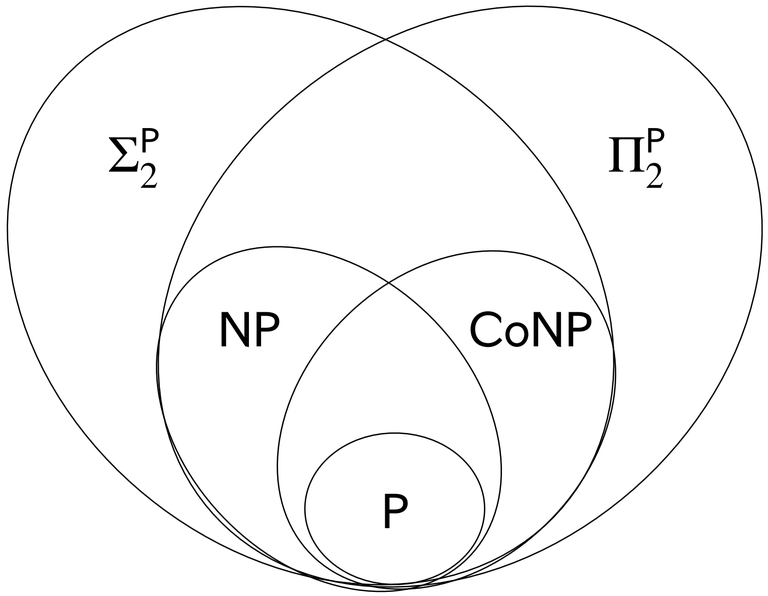

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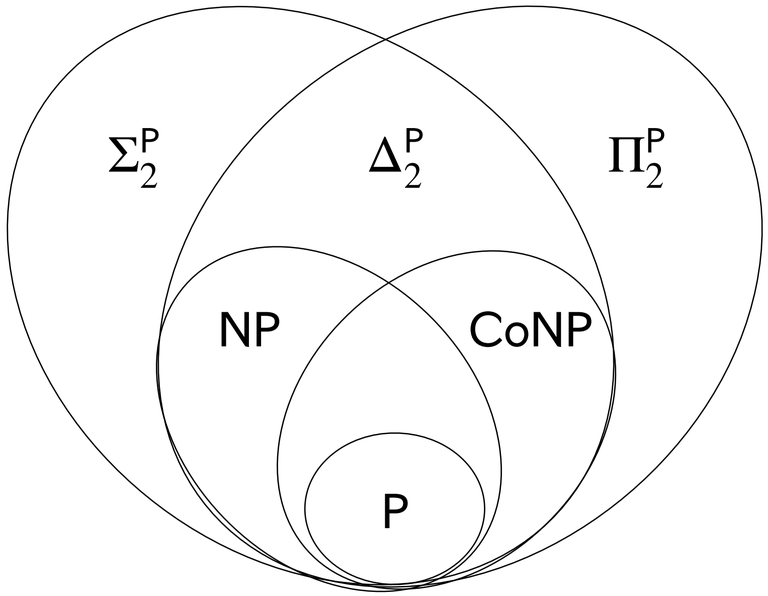

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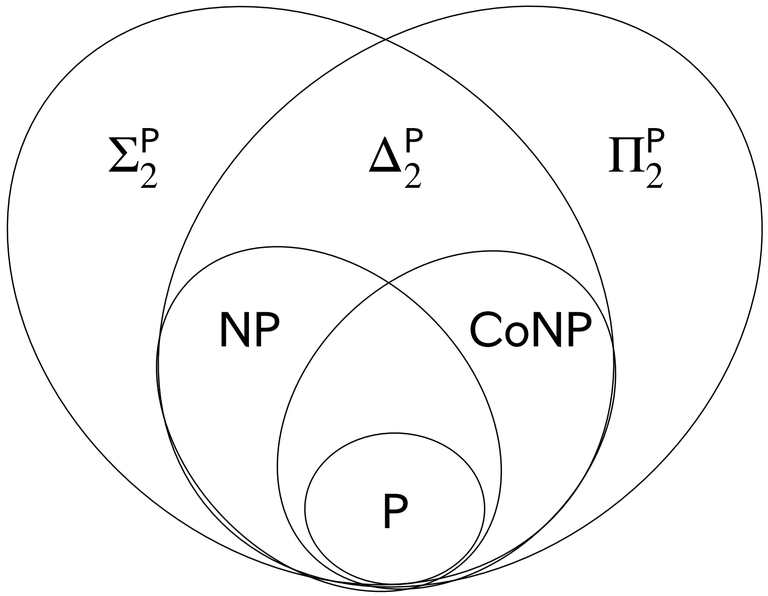

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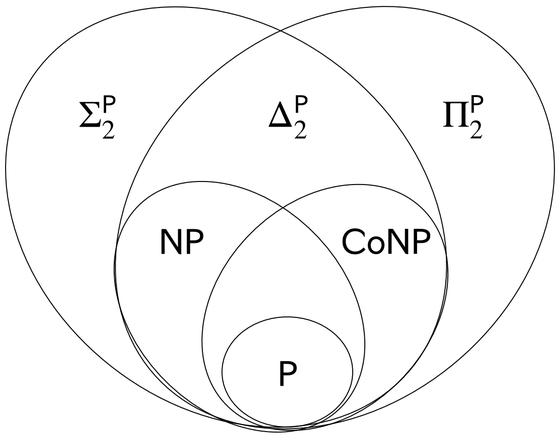

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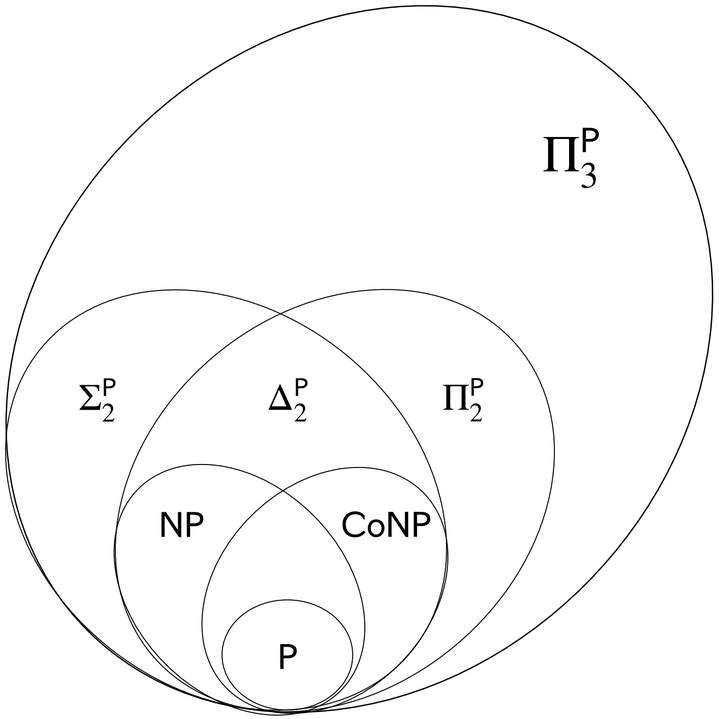

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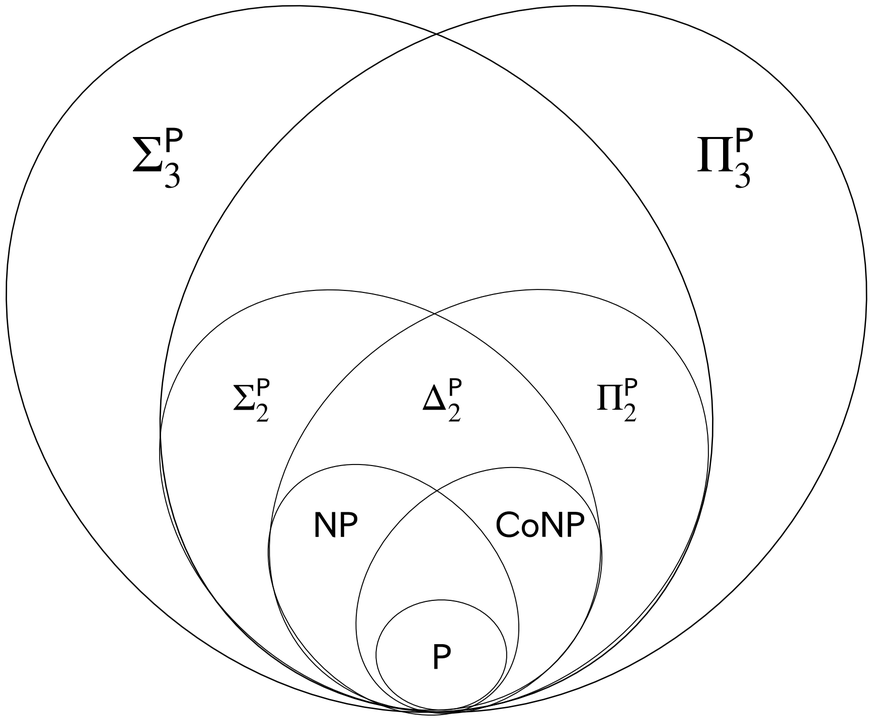

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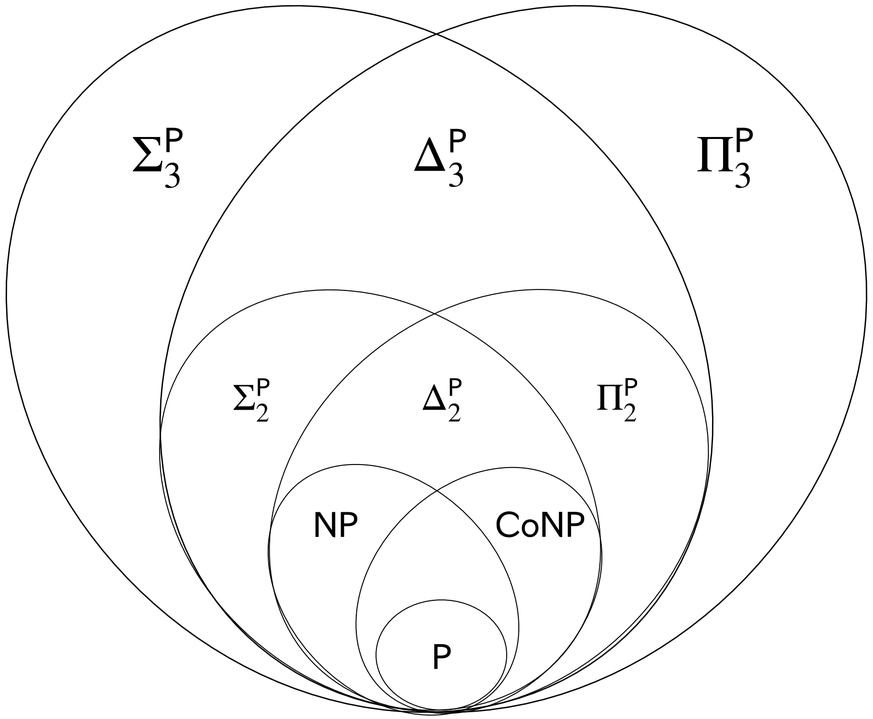

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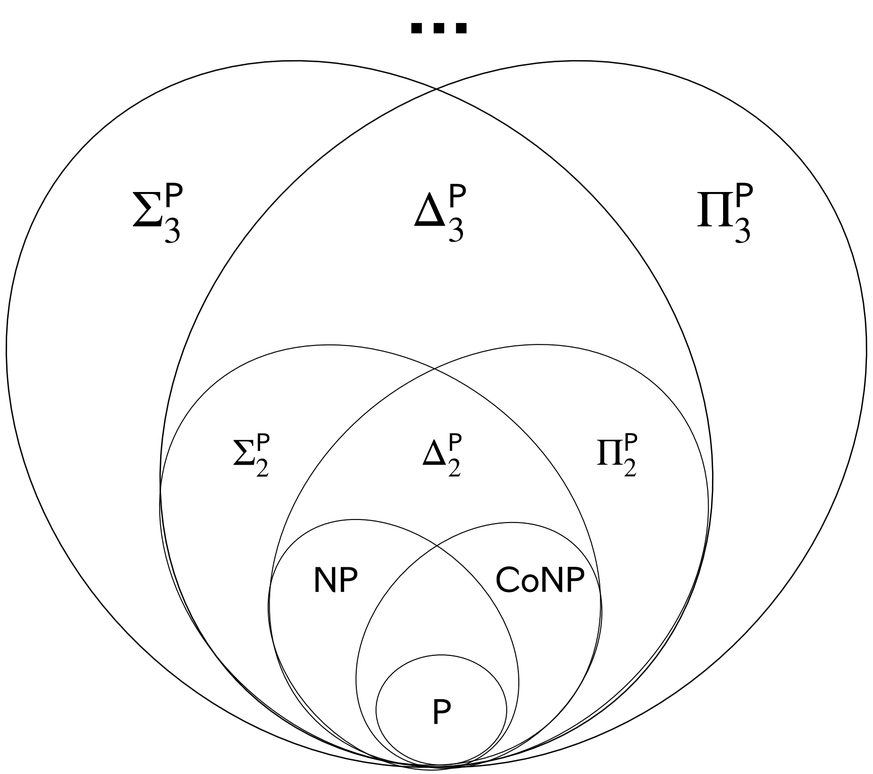

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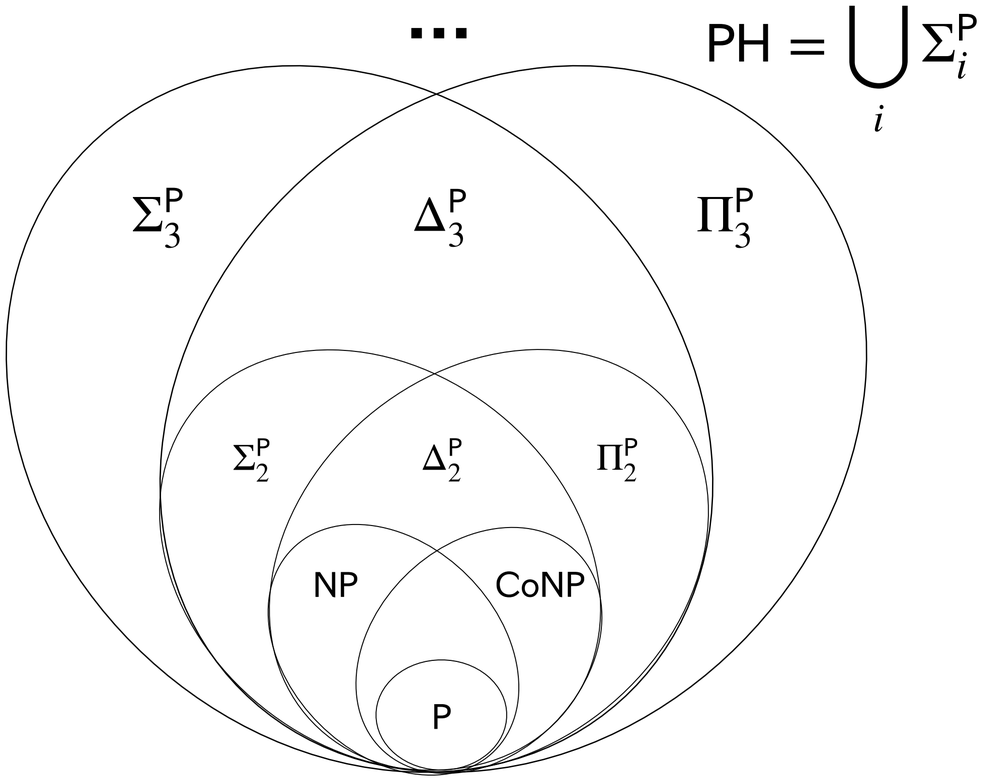

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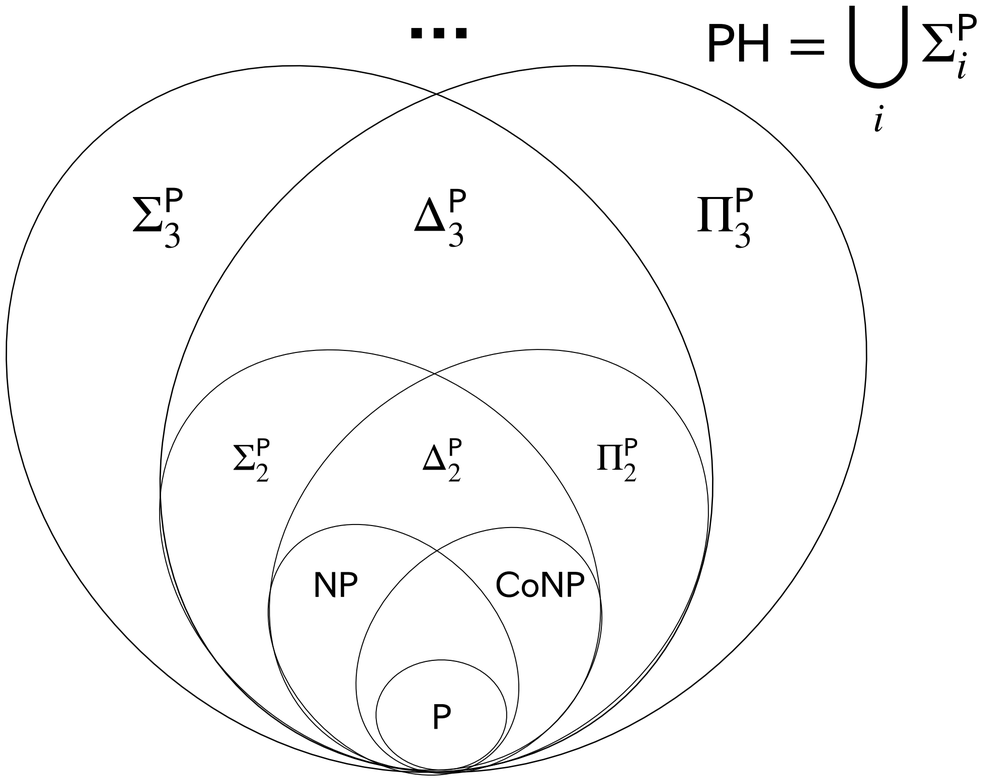

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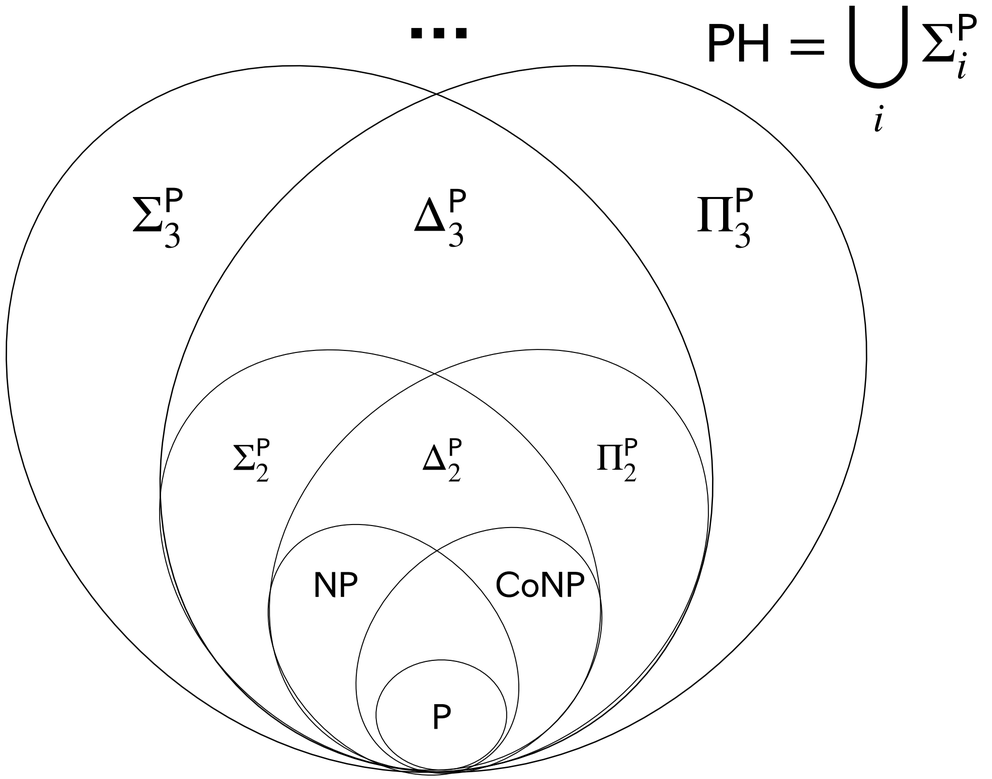

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#SAT **#P-complete** even for

- monotone 2CNF and
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Applications and Solvers

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Probabilistic Inference

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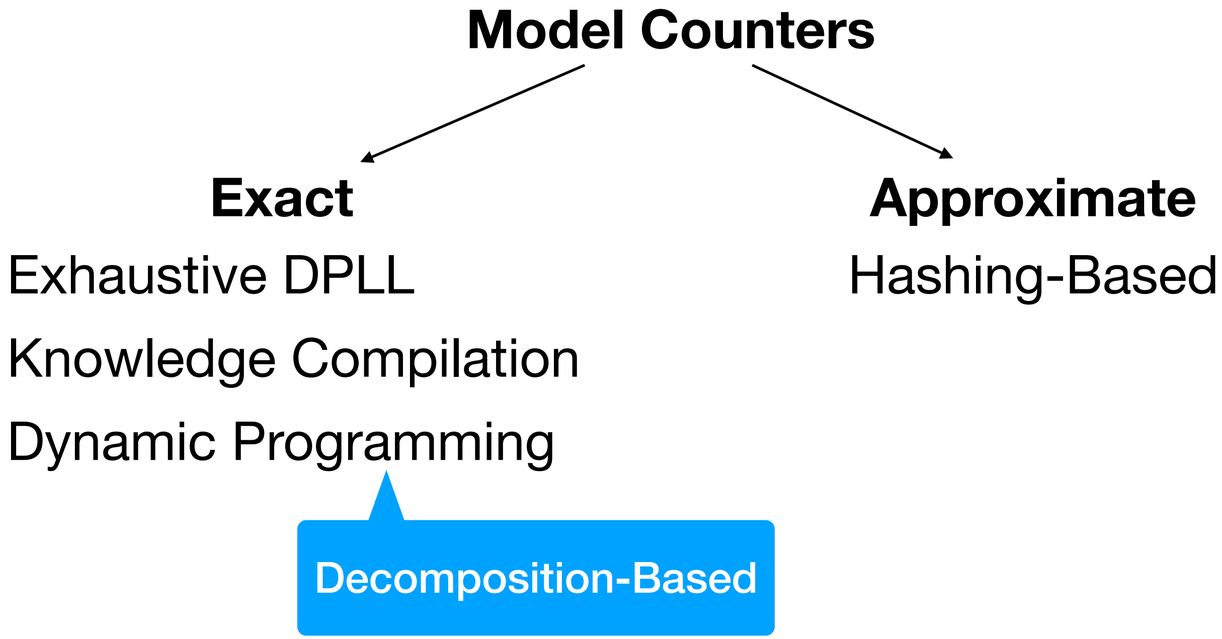

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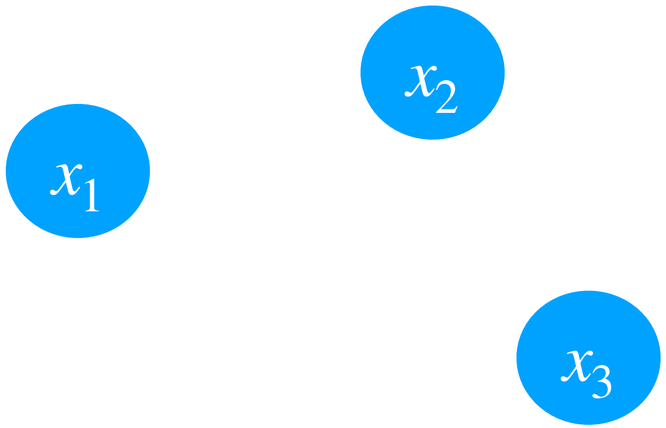

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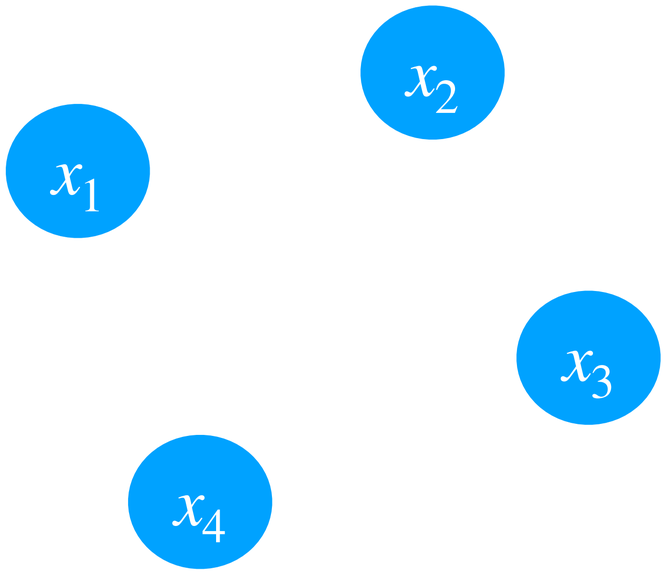

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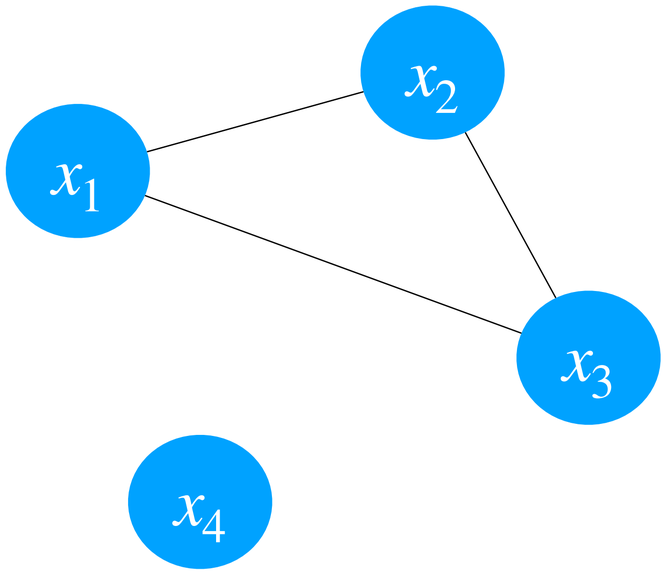

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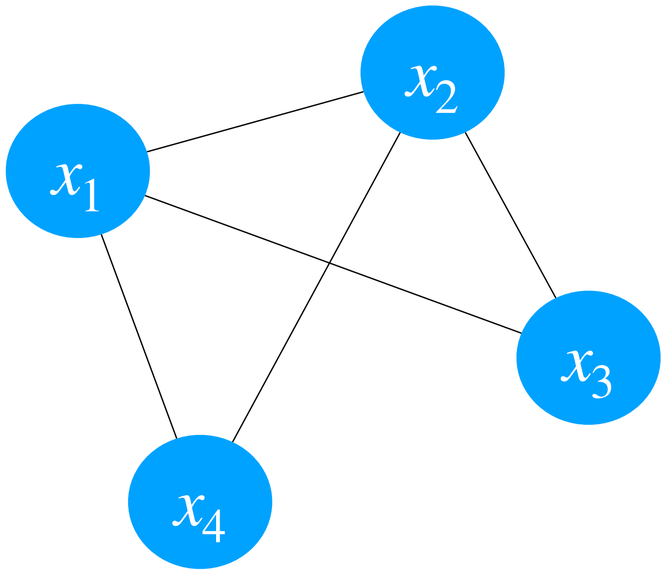

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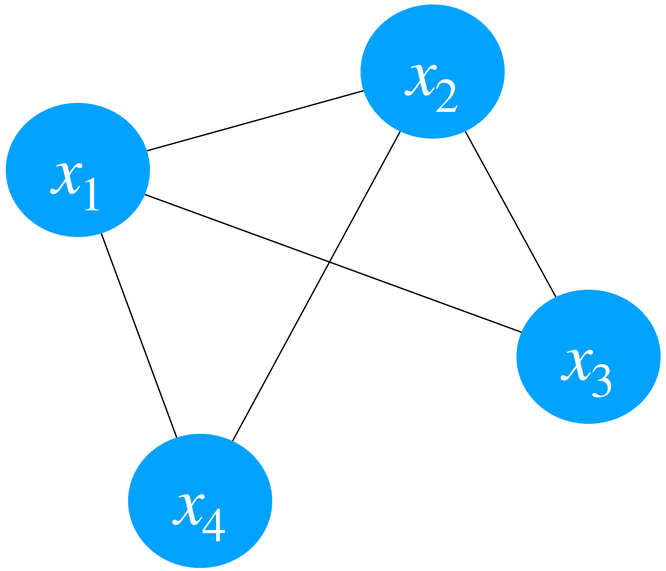

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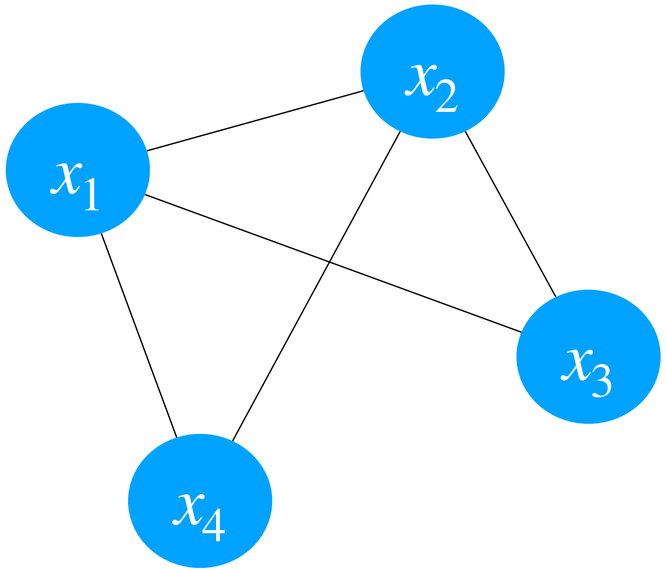

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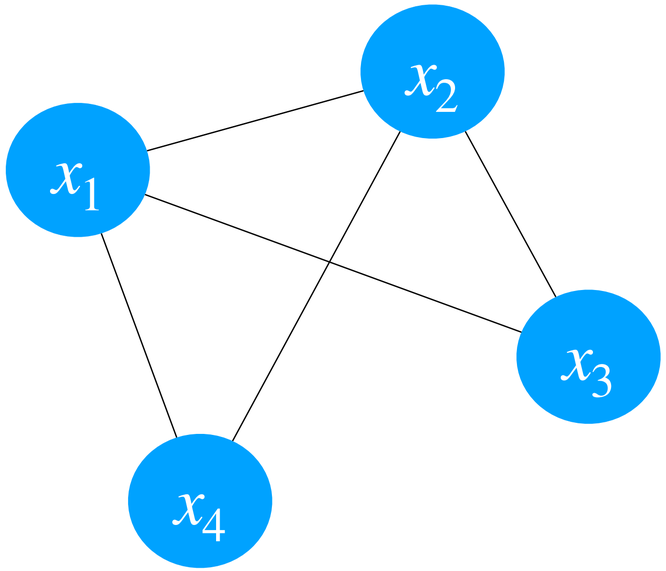

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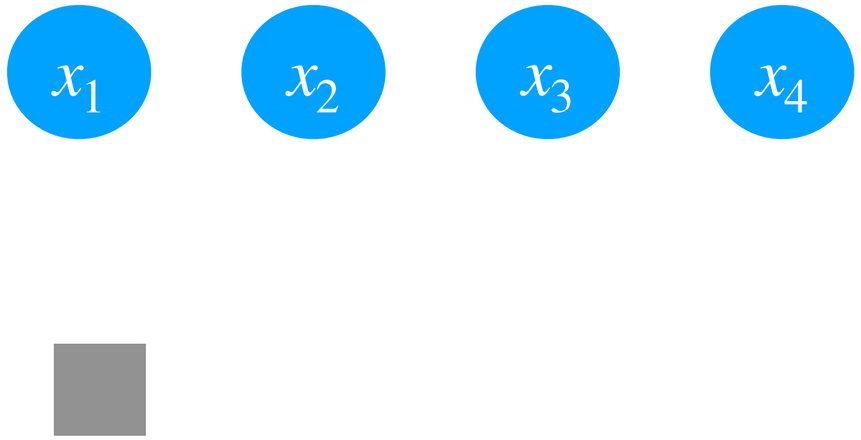

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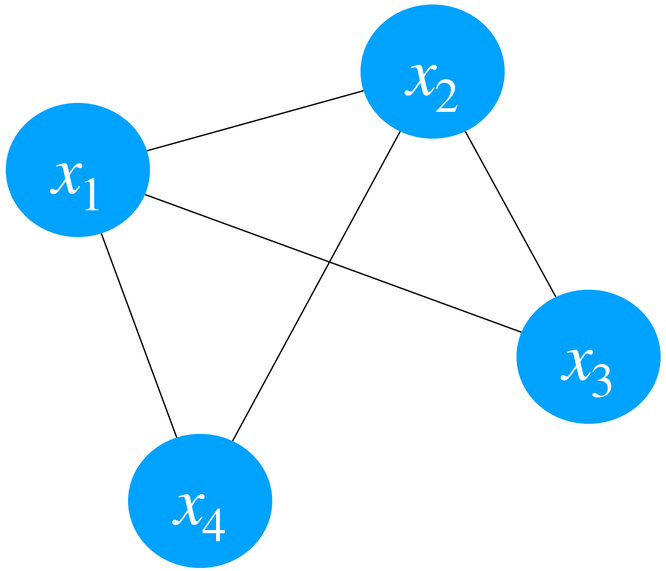

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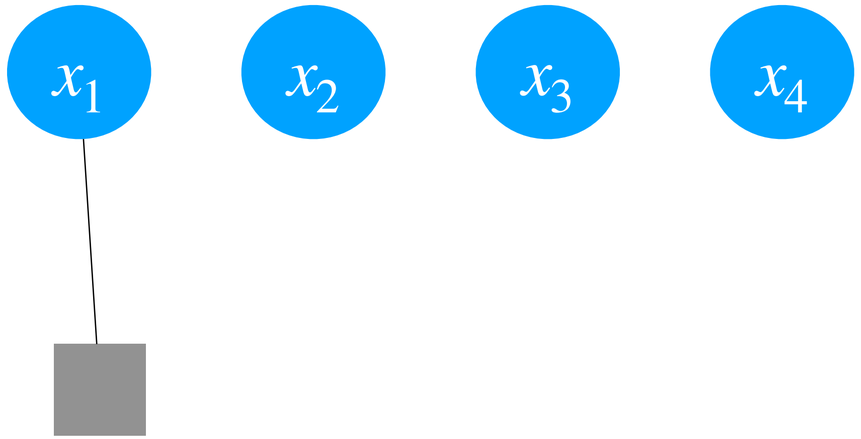

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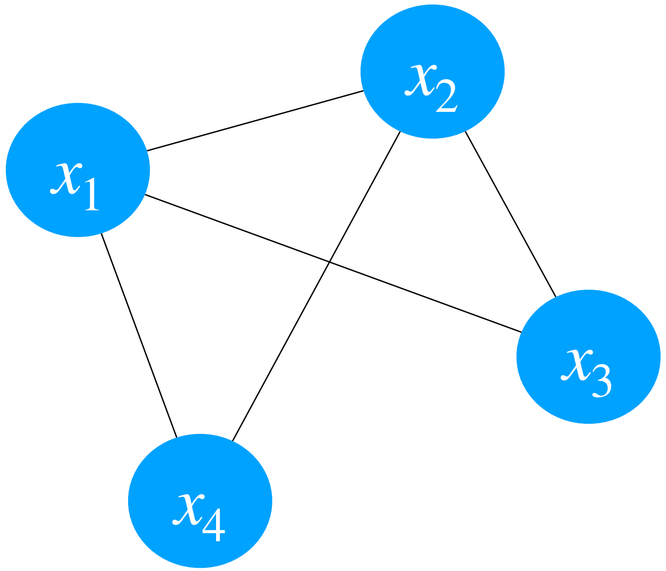

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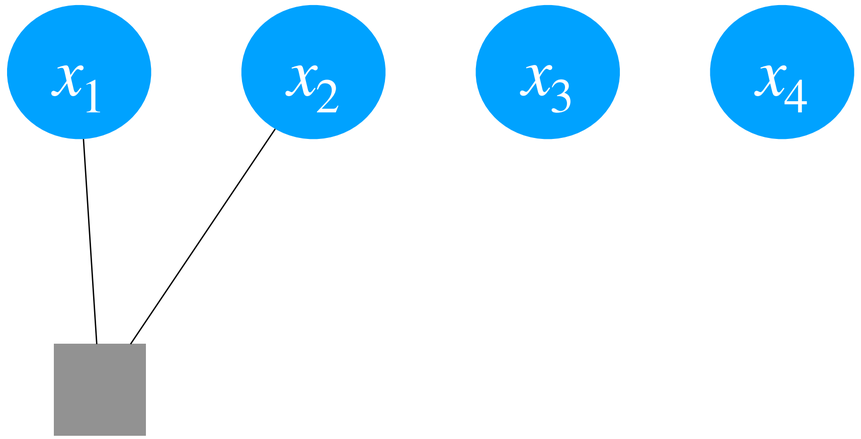

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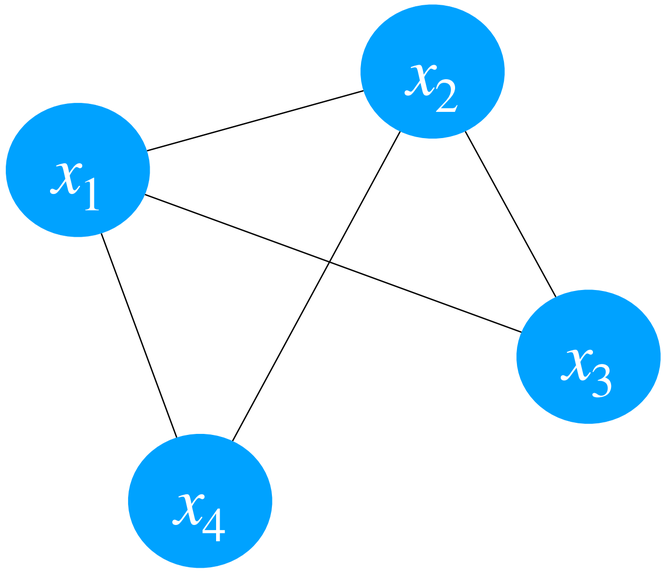

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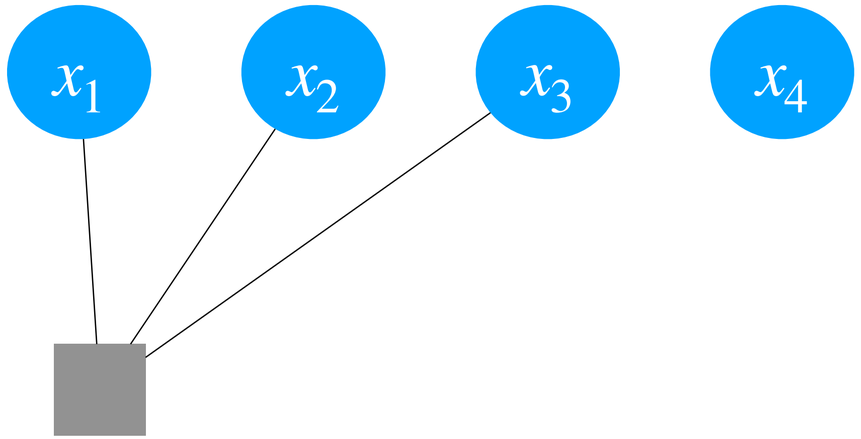

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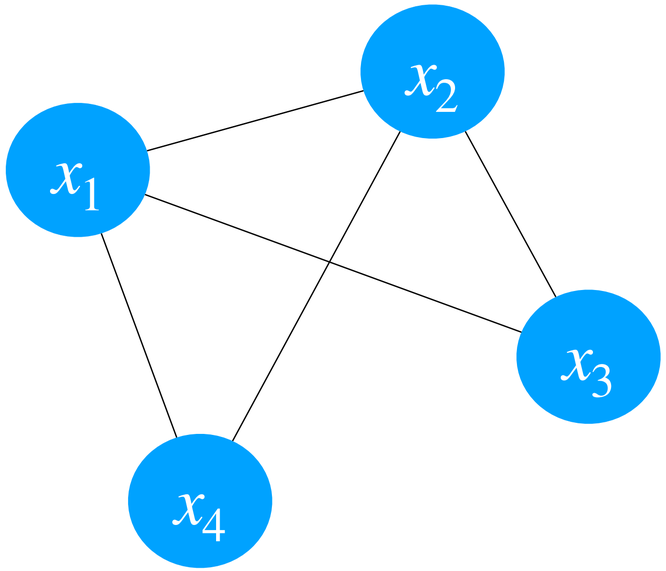

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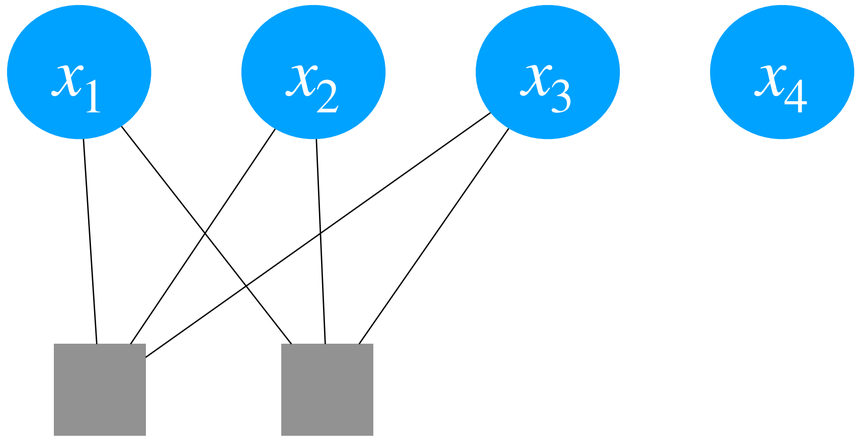

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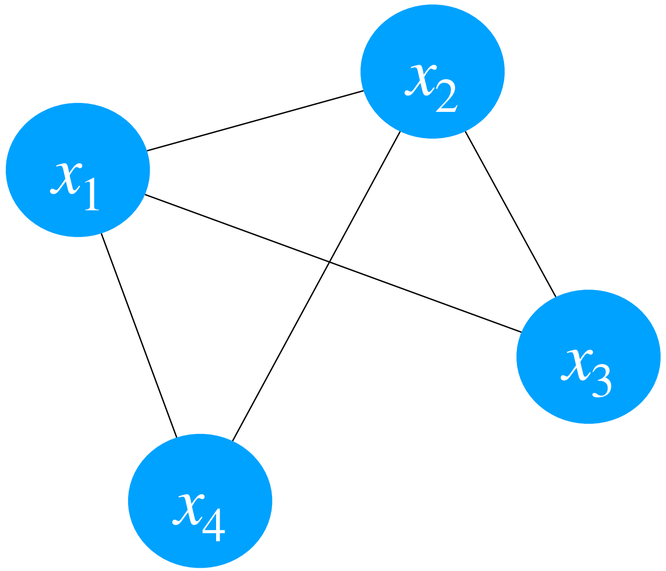

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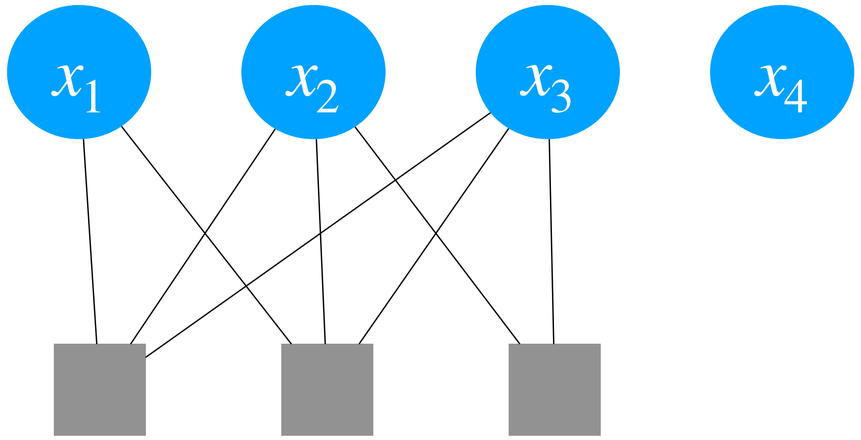

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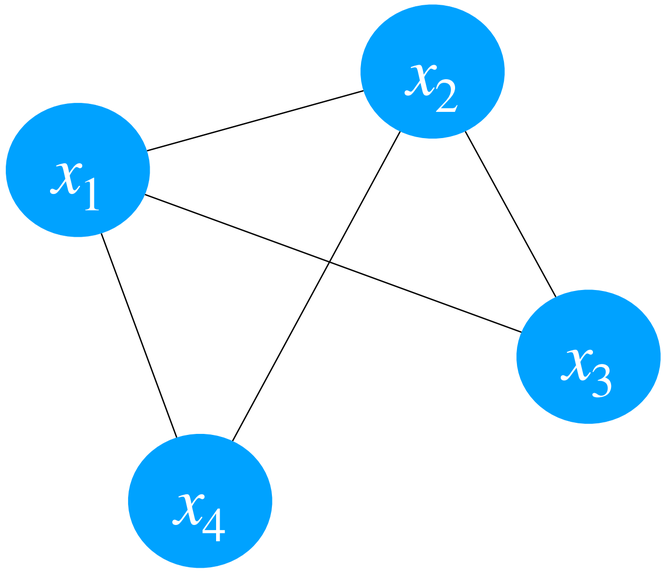

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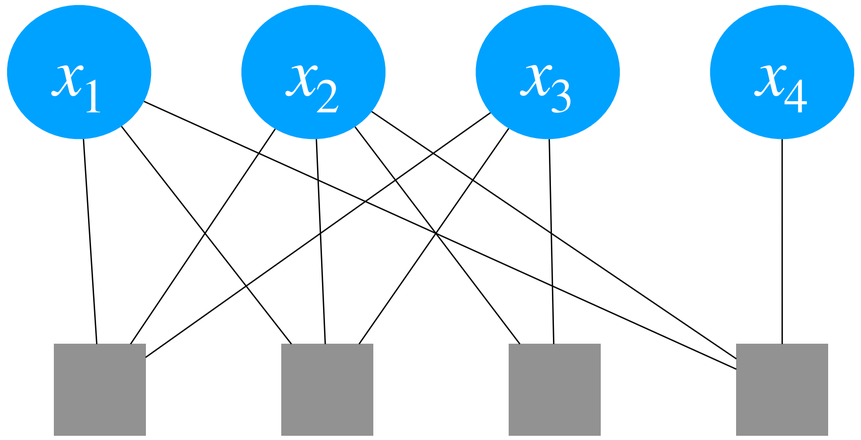

#SAT and Treewidth

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primal treewidth k

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incidence treewidth k^*

primal treewidth k

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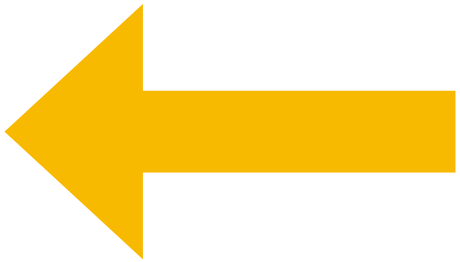
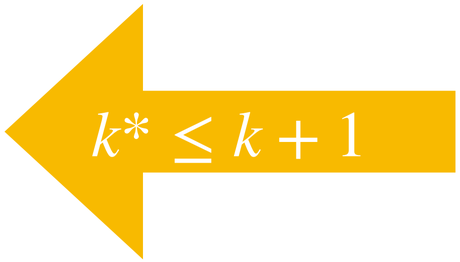
incidence treewidth k^*  **primal treewidth k**

#SAT and Treewidth

incidence treewidth k^*  **primal treewidth** k

#SAT and Treewidth

incidence treewidth k^*

$k^* \leq k + 1$

primal treewidth k

$f(k) \cdot p(F)$

#SAT and Treewidth

incidence treewidth k^* ← $k^* \leq k + 1$ → **primal treewidth k**

$f(k^*) \cdot p(F)$ — Courcelle, Makowsky, Rotics 2001

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Samer and Szeider 2007

Our result

$2^{k^*} \cdot p(F)$

Samer and Szeider 2007

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Fischer, Makowsky, Ravve 2006
Samer and Szeider 2007

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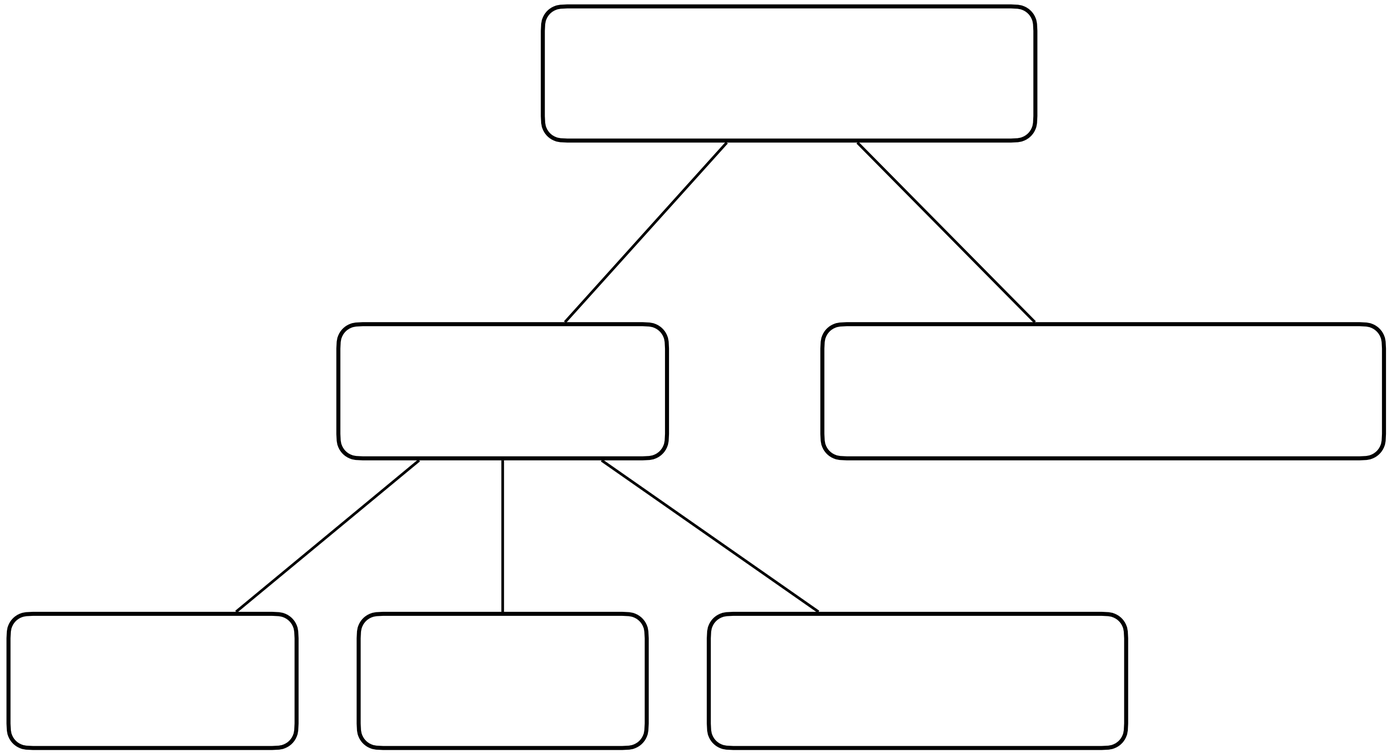
Optimal under ETH

Samer and Szeider 2007

$2^k \cdot p(F)$

Tree Decompositions

Tree Decompositions



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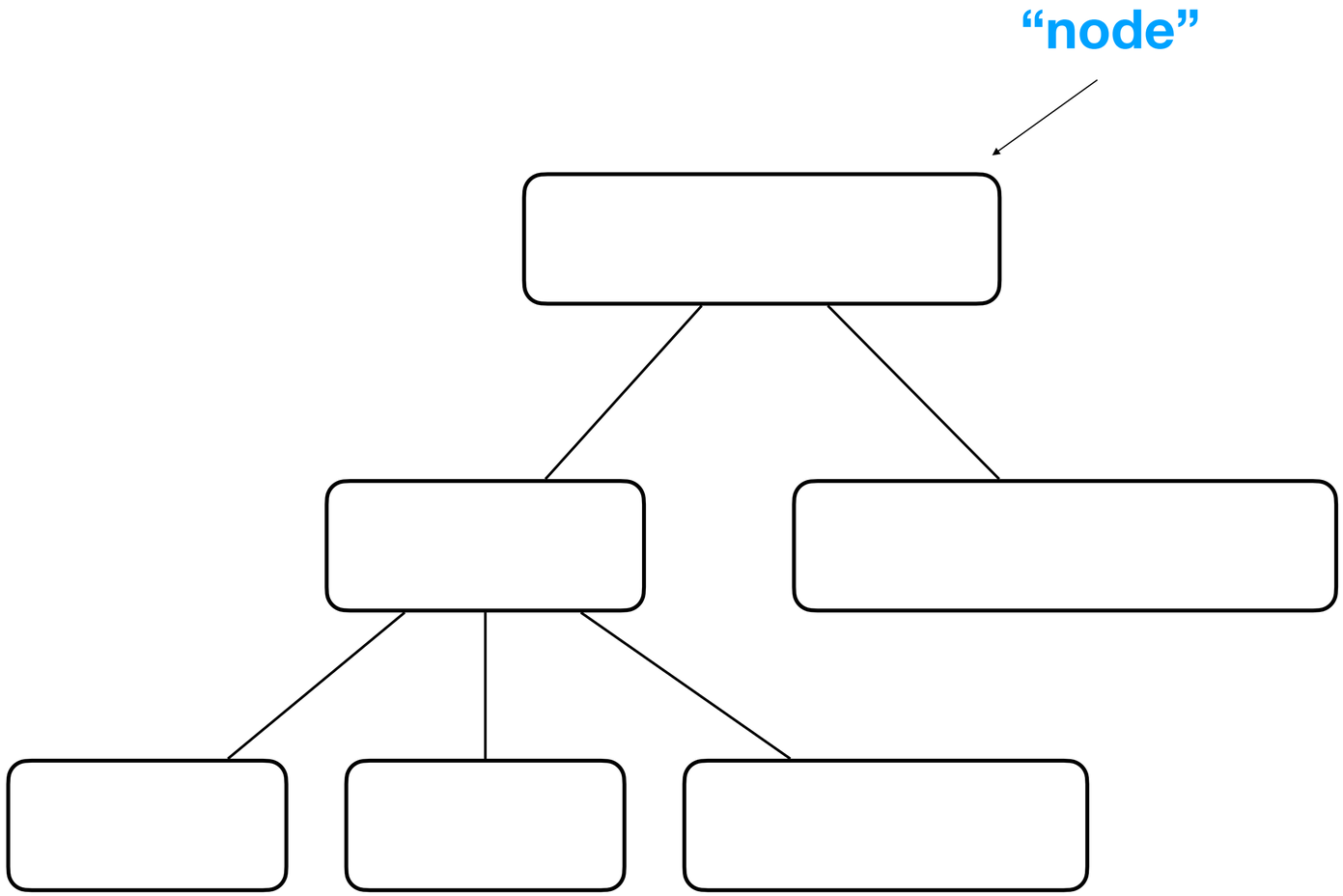

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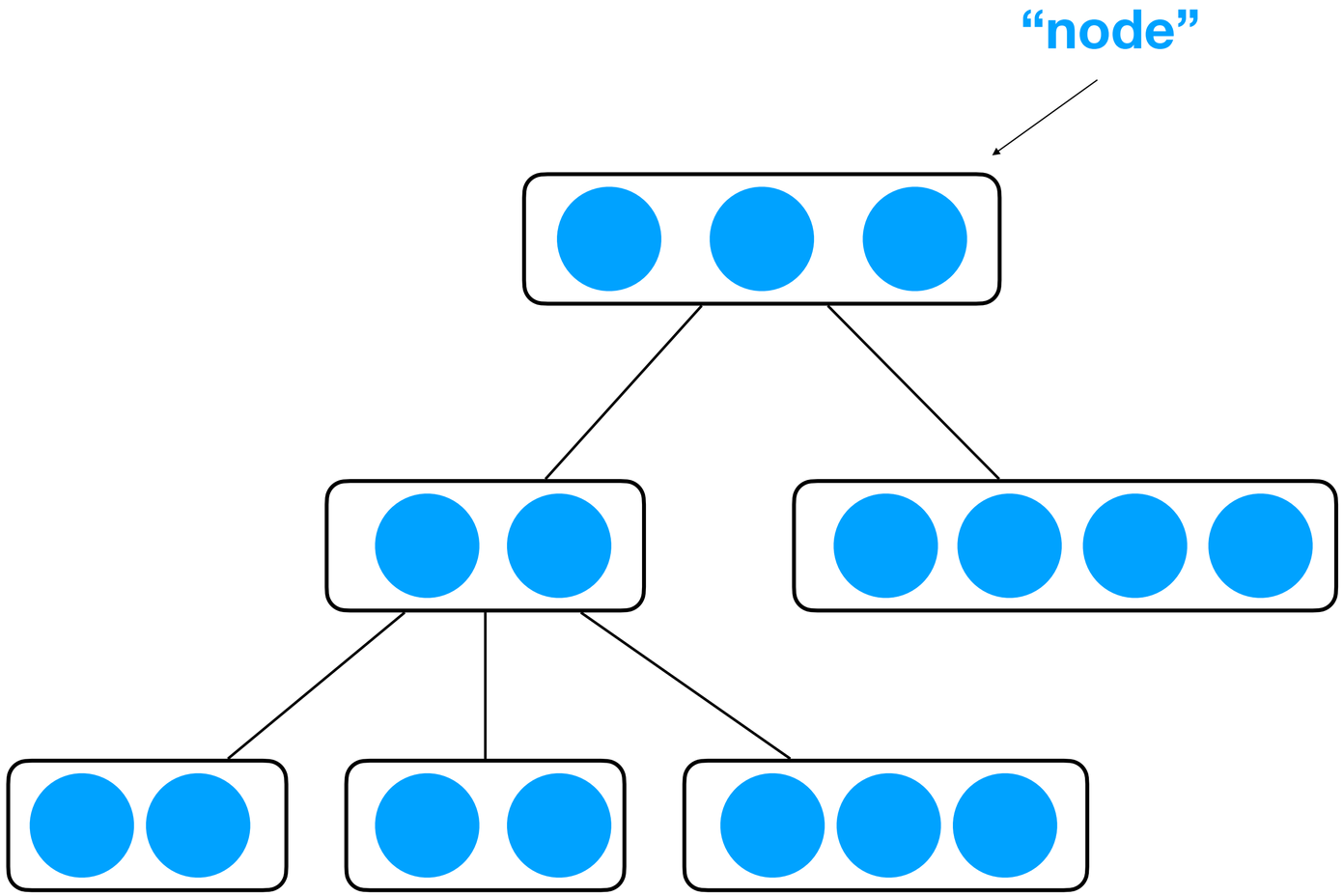

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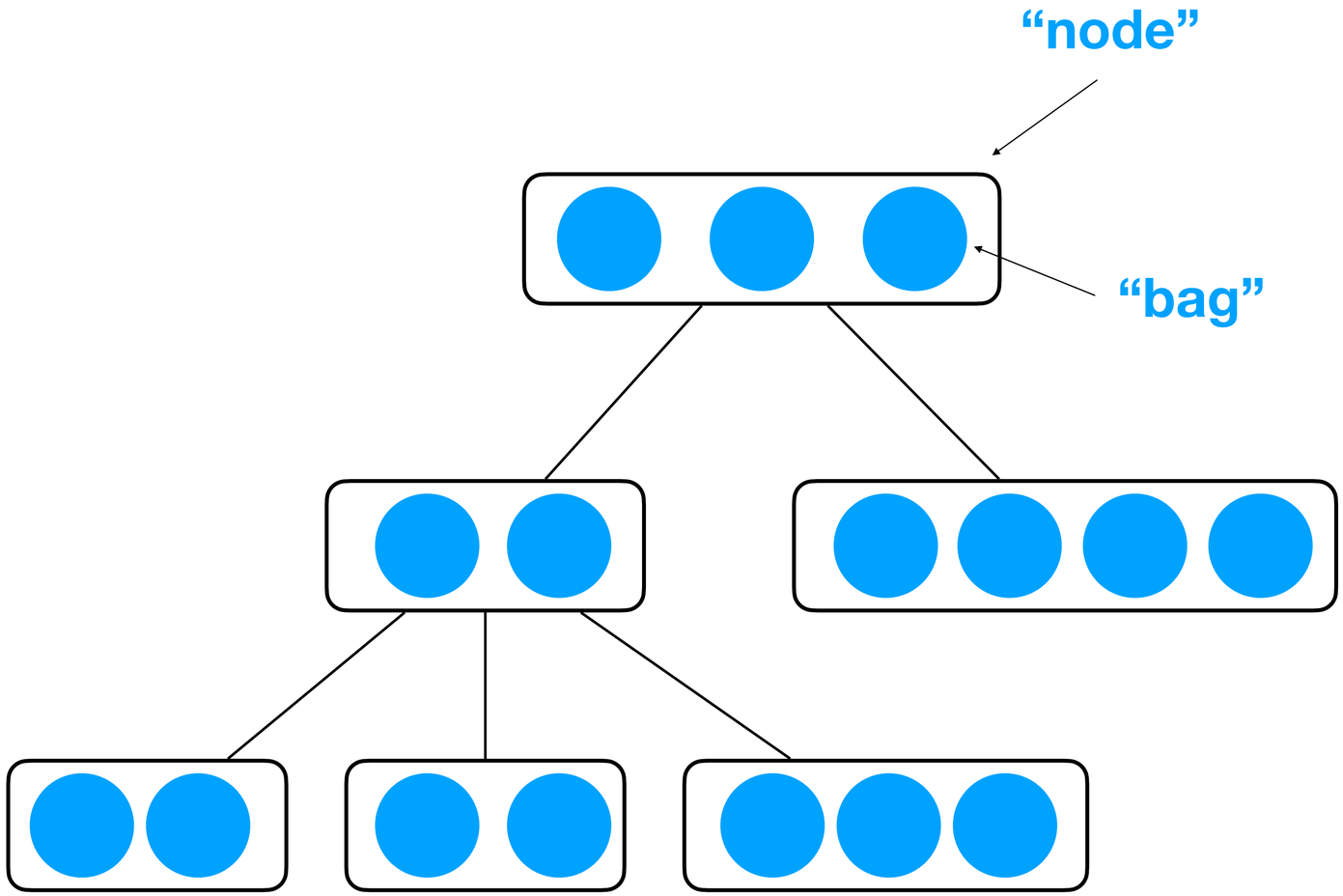
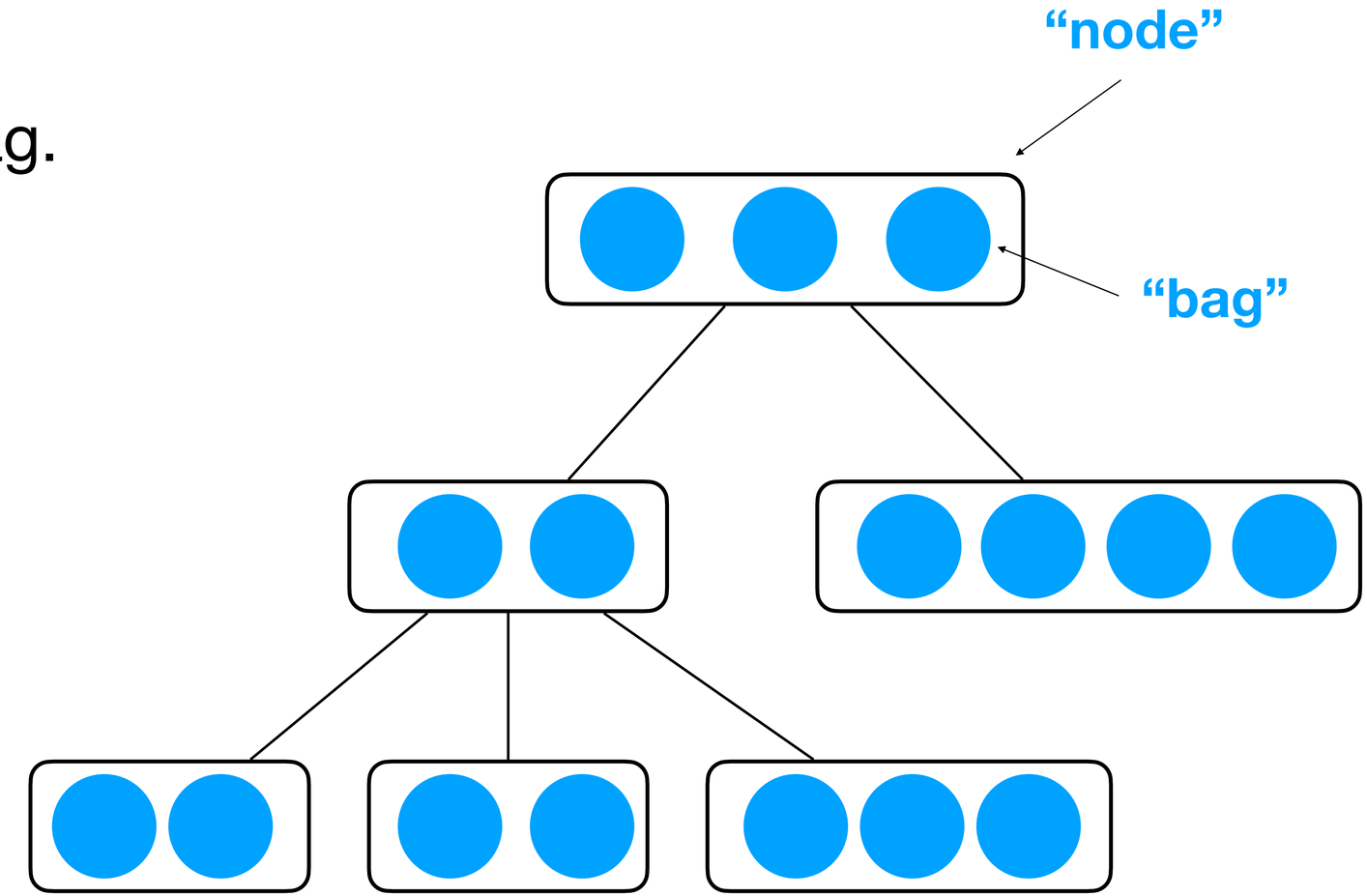


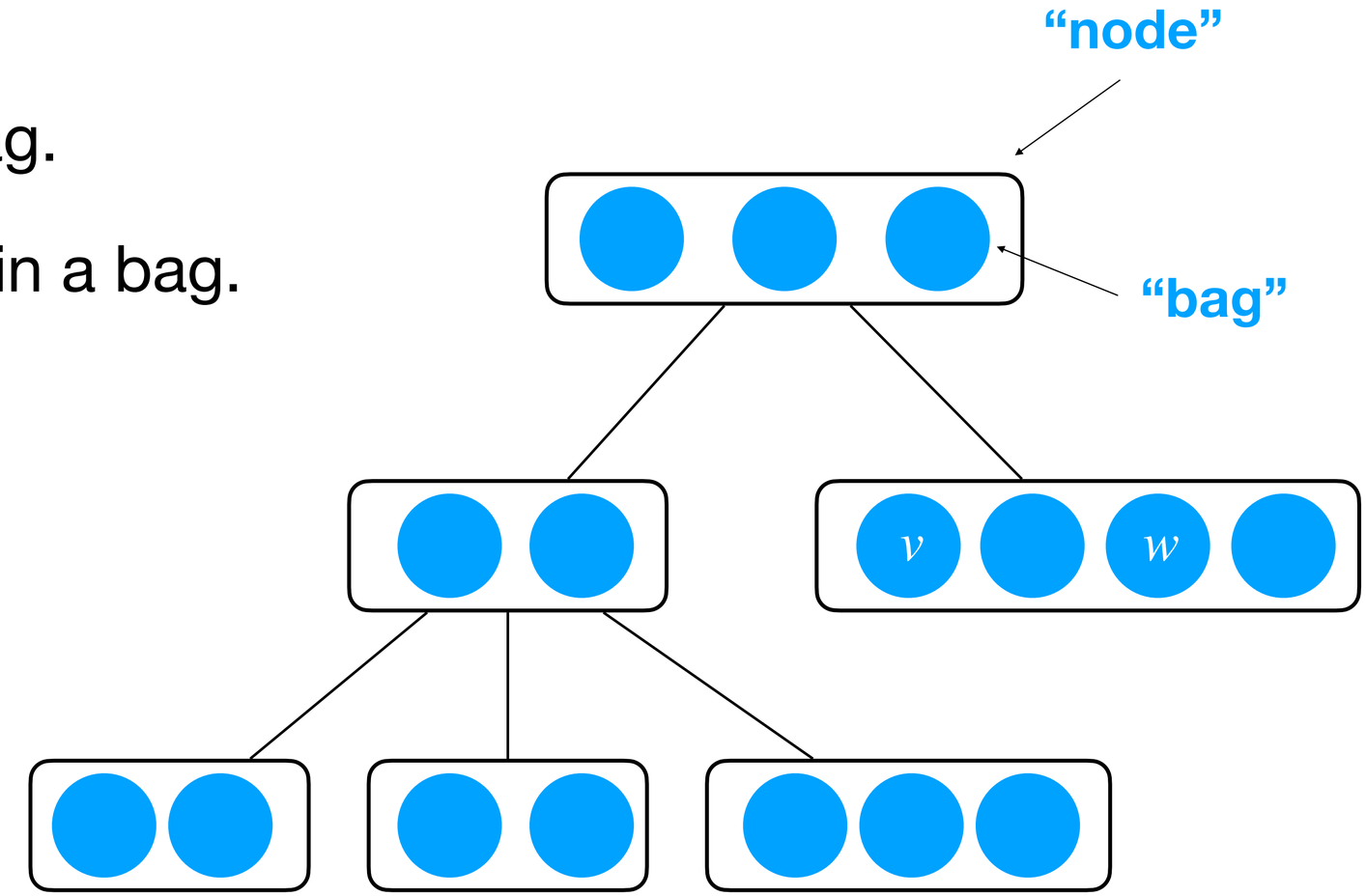
Tree Decompositions

1. Each **vertex** appears in a bag.



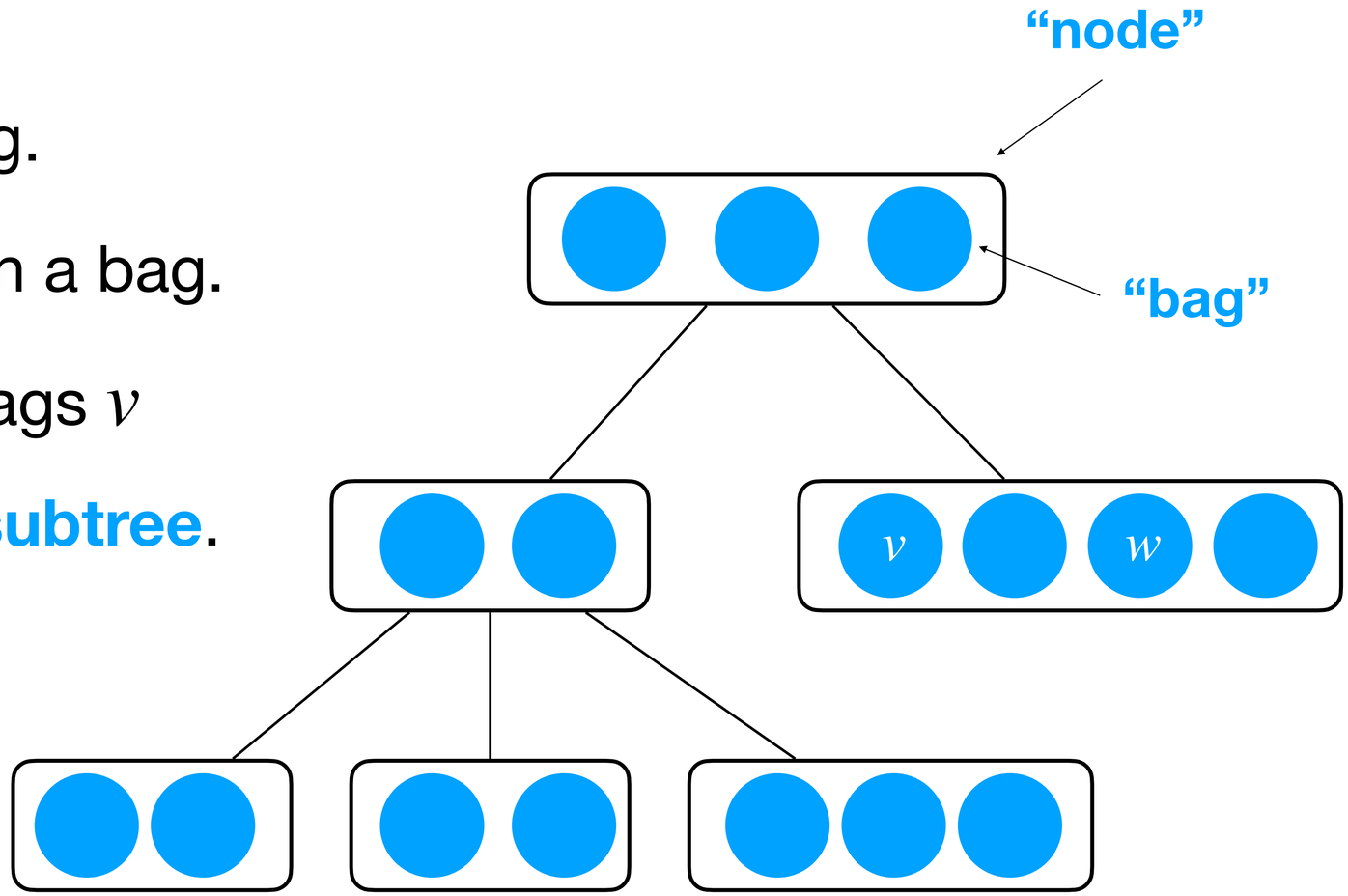
Tree Decompositions

1. Each **vertex** appears in a bag.
2. Each **edge** vw is contained in a bag.



Tree Decompositions

1. Each **vertex** appears in a bag.
2. Each **edge** vw is contained in a bag.
3. The set of nodes in whose bags v appears form a **connected subtree**.



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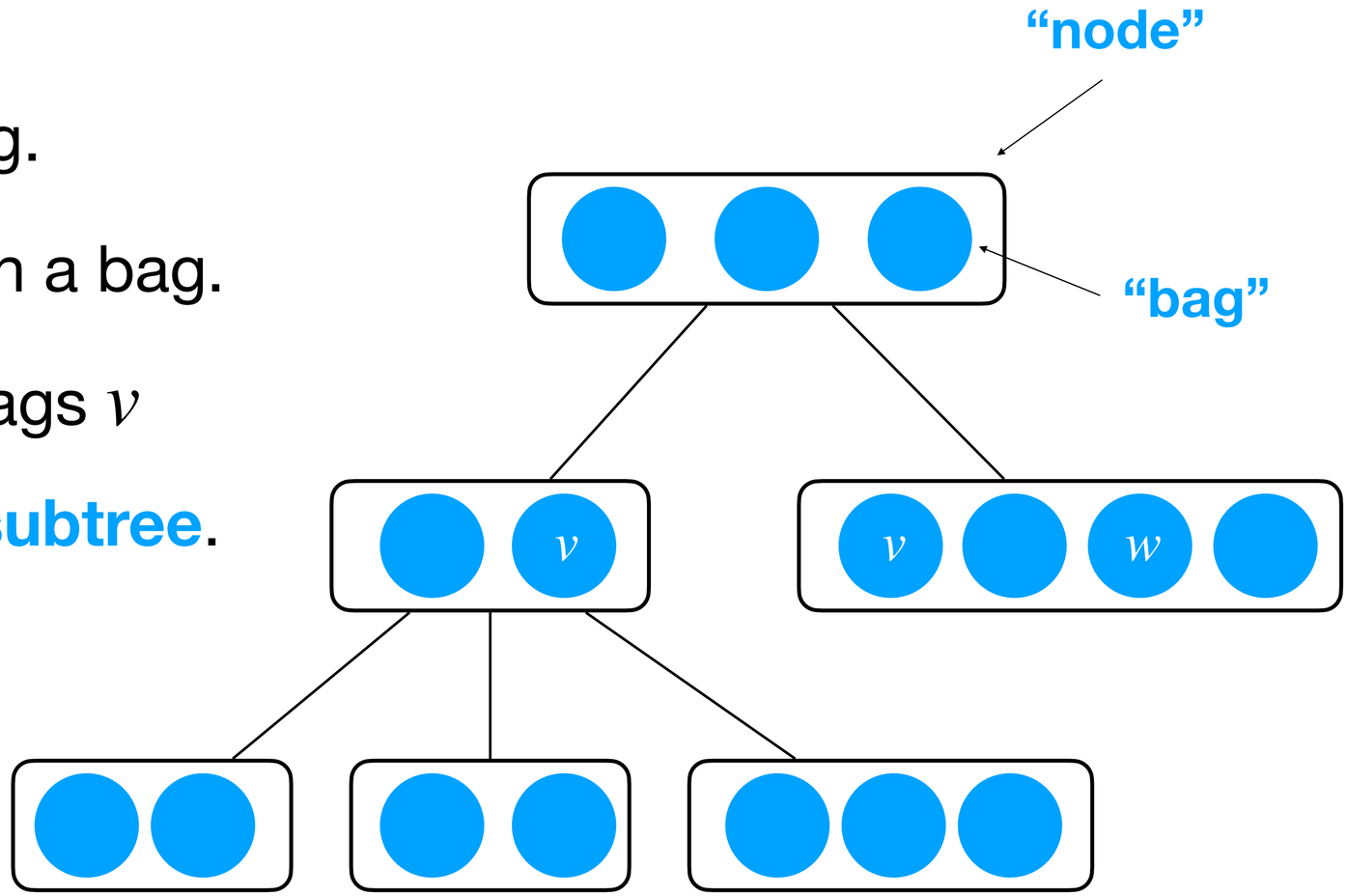

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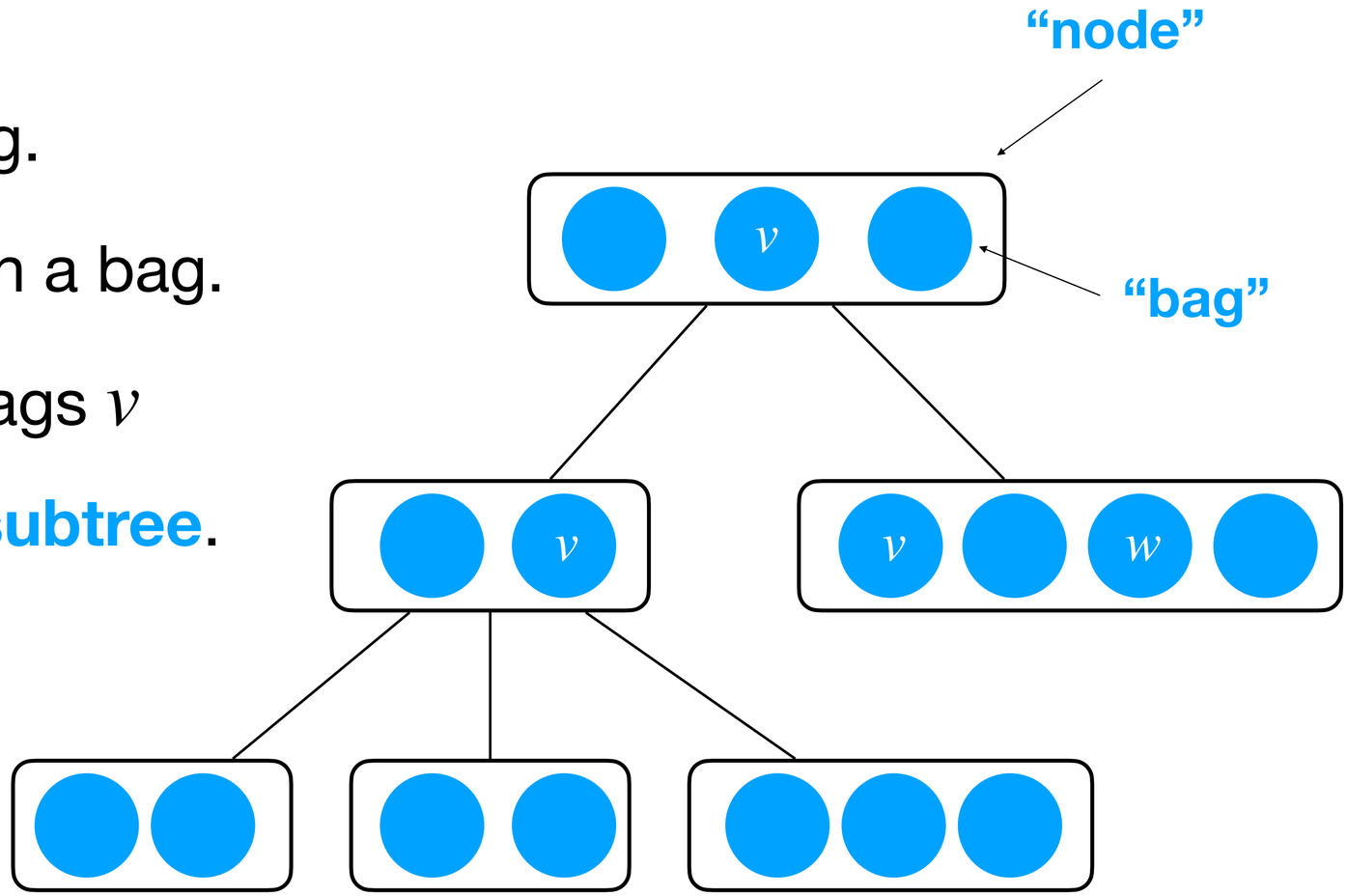

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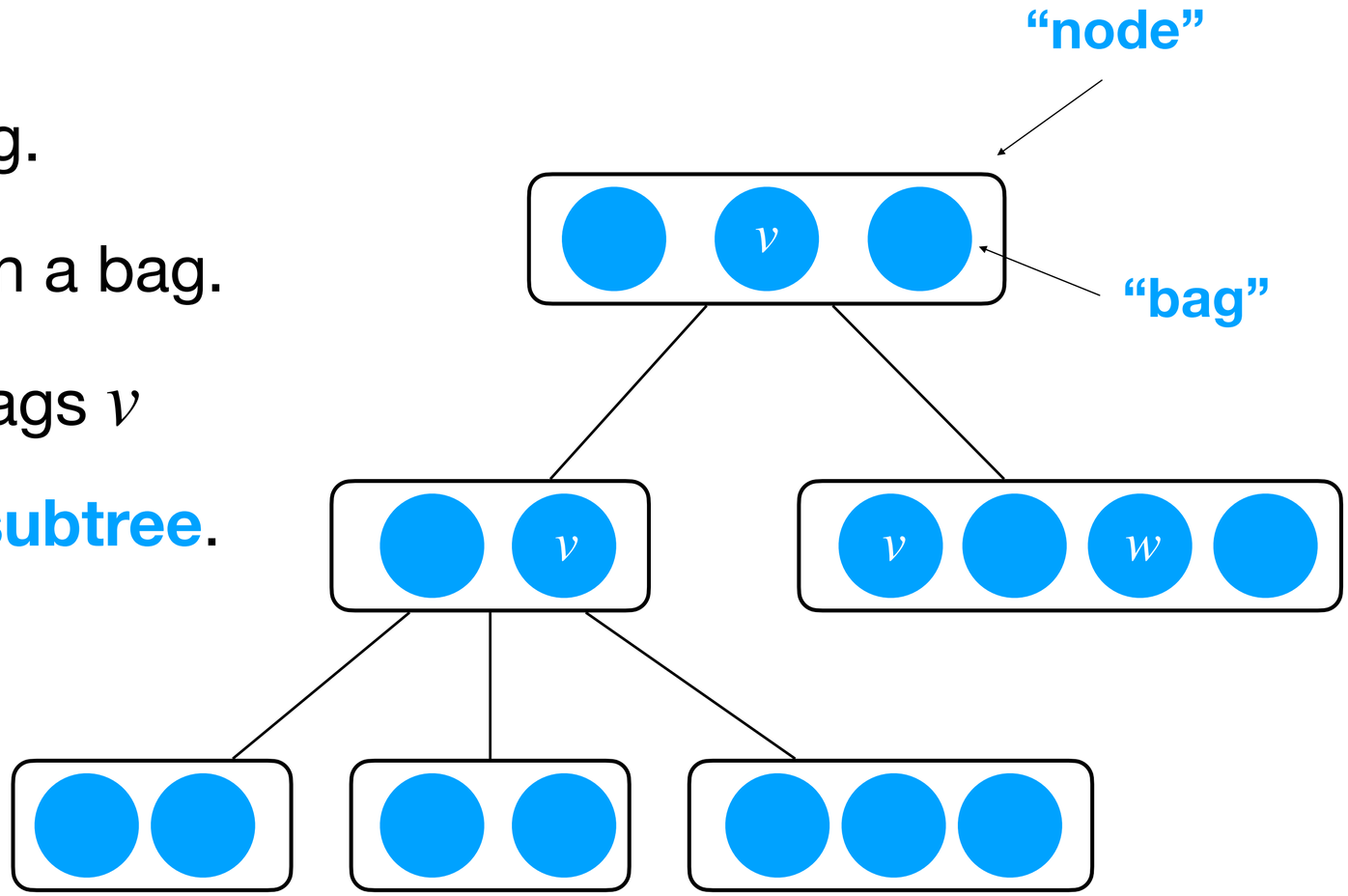
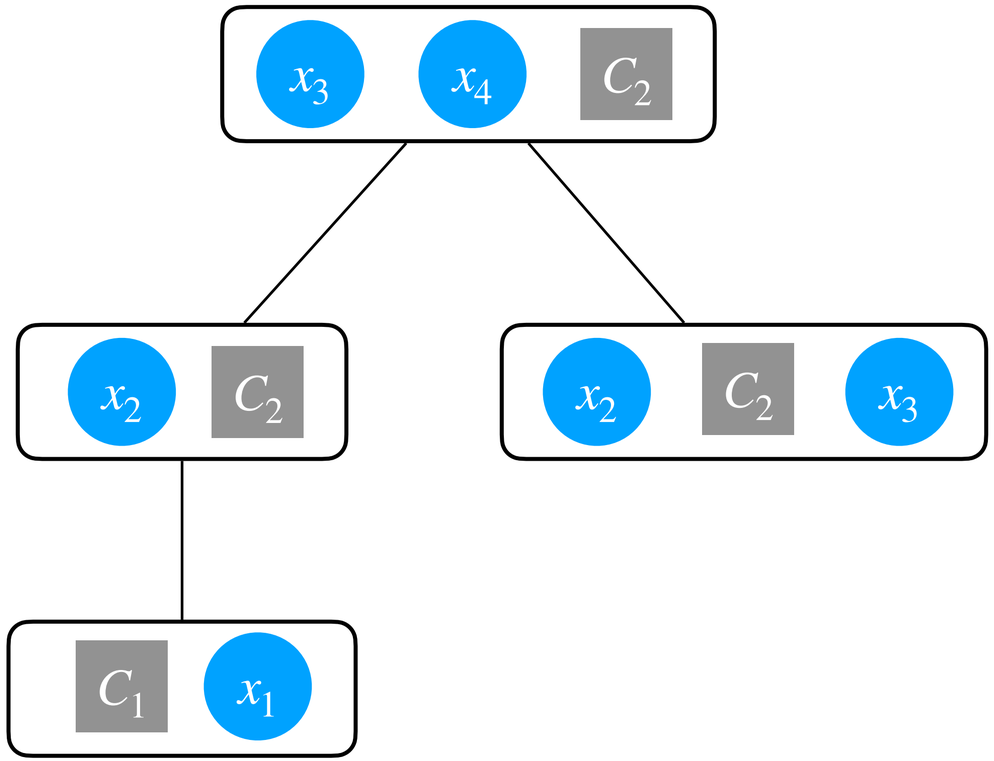


The **width** of a tree decomposition is the size of its largest bag - 1.

Dynamic Programming for #SAT



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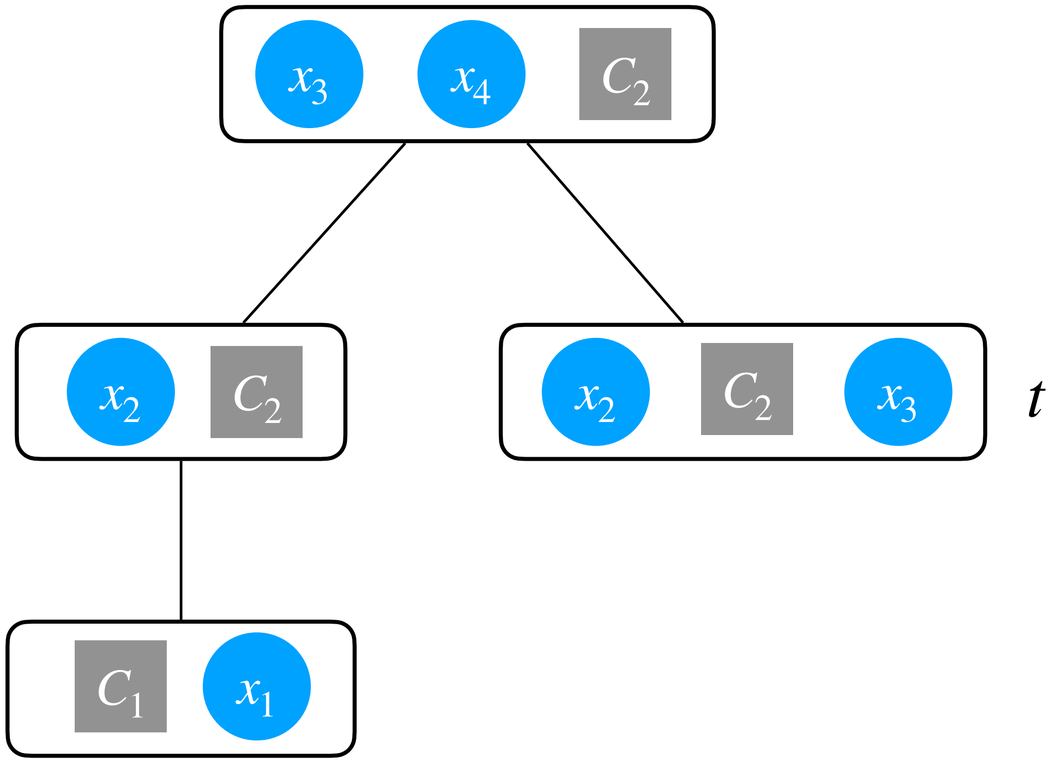

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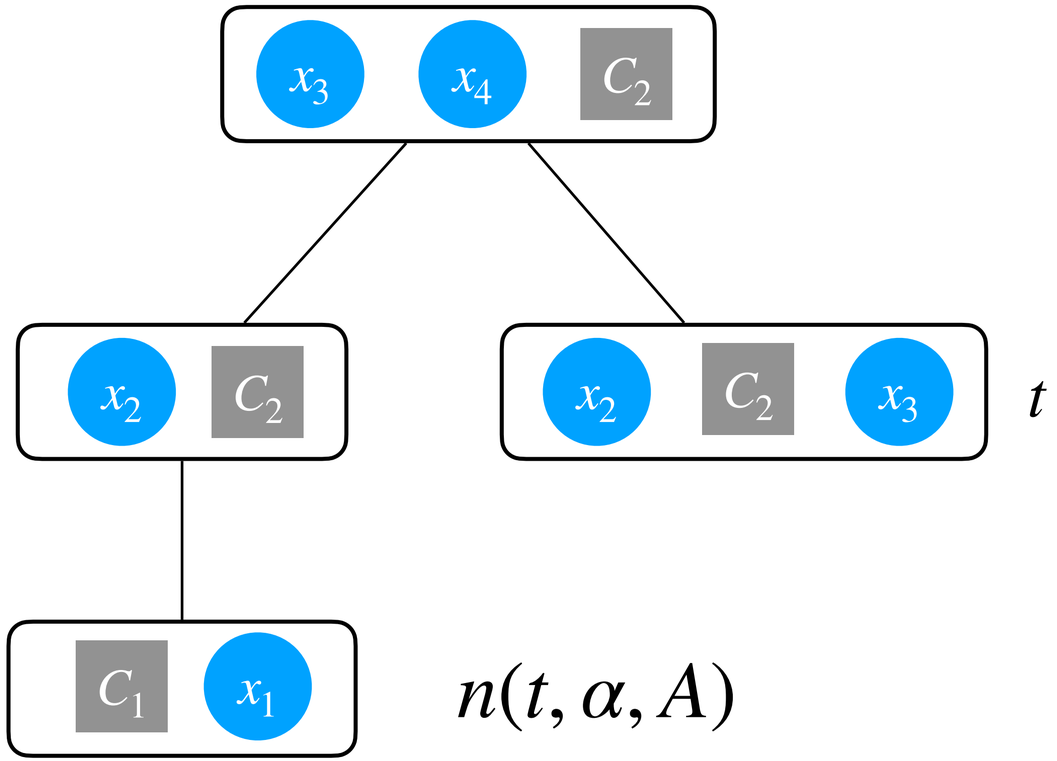

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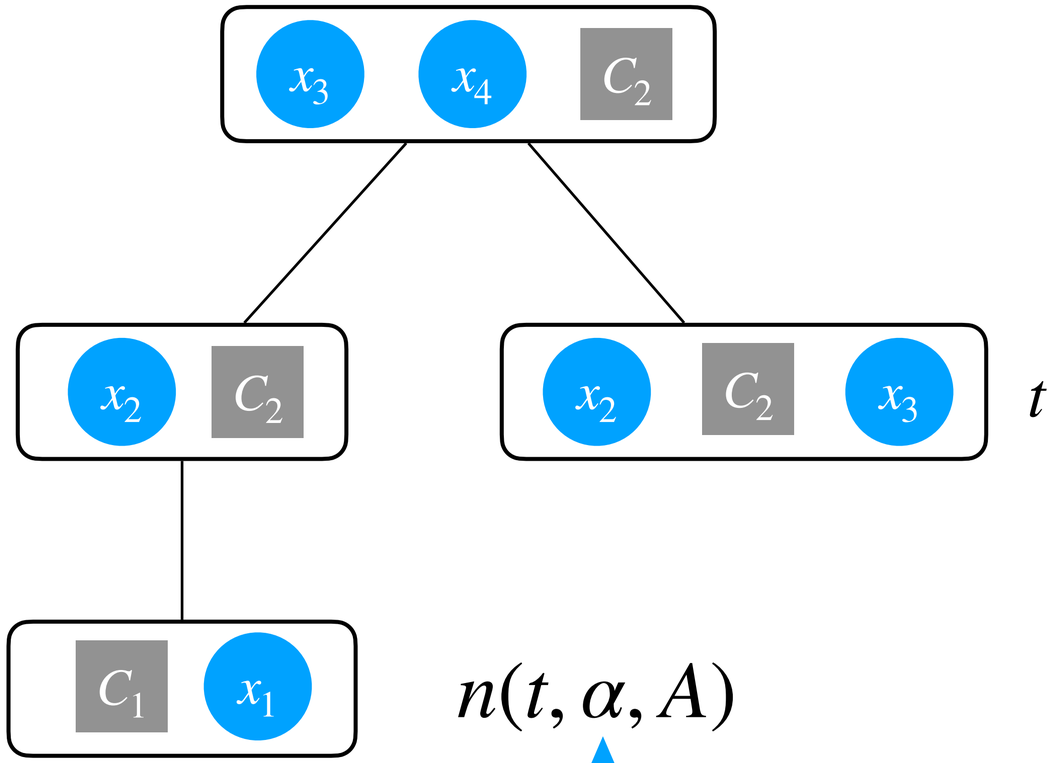

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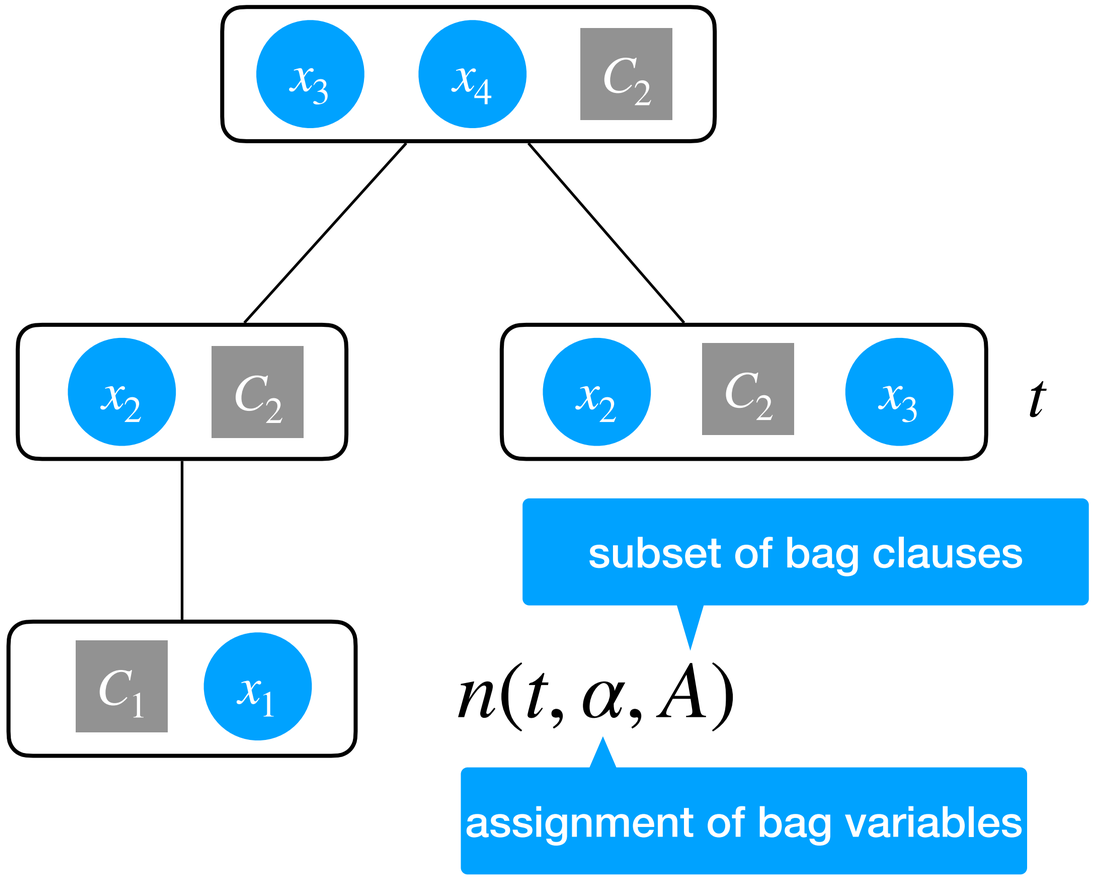
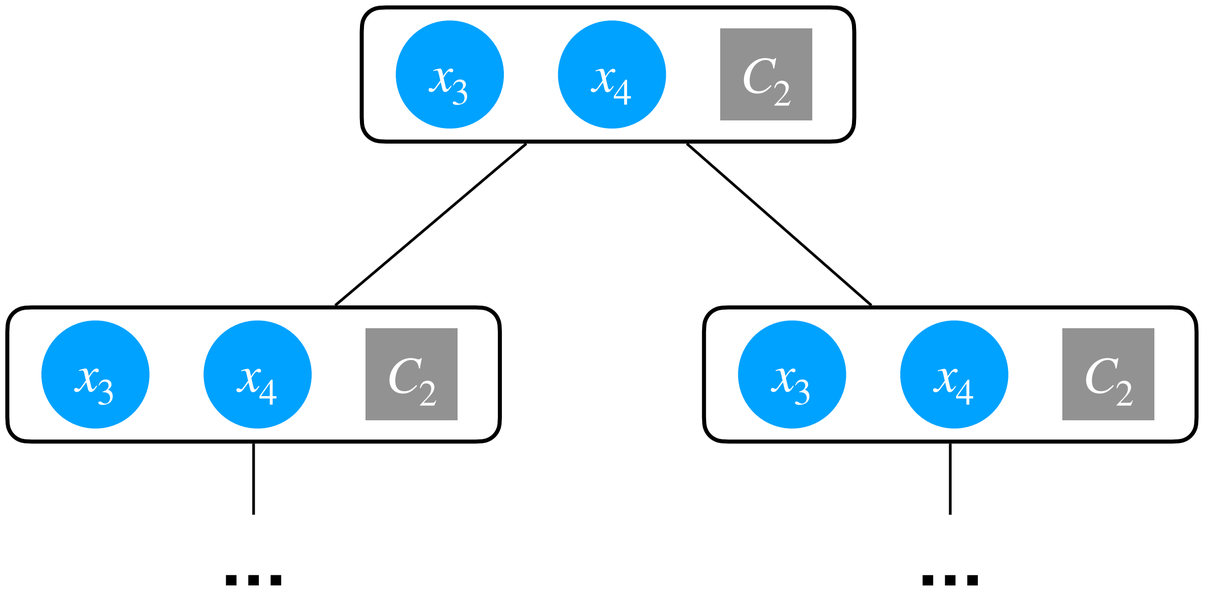


Join Nodes



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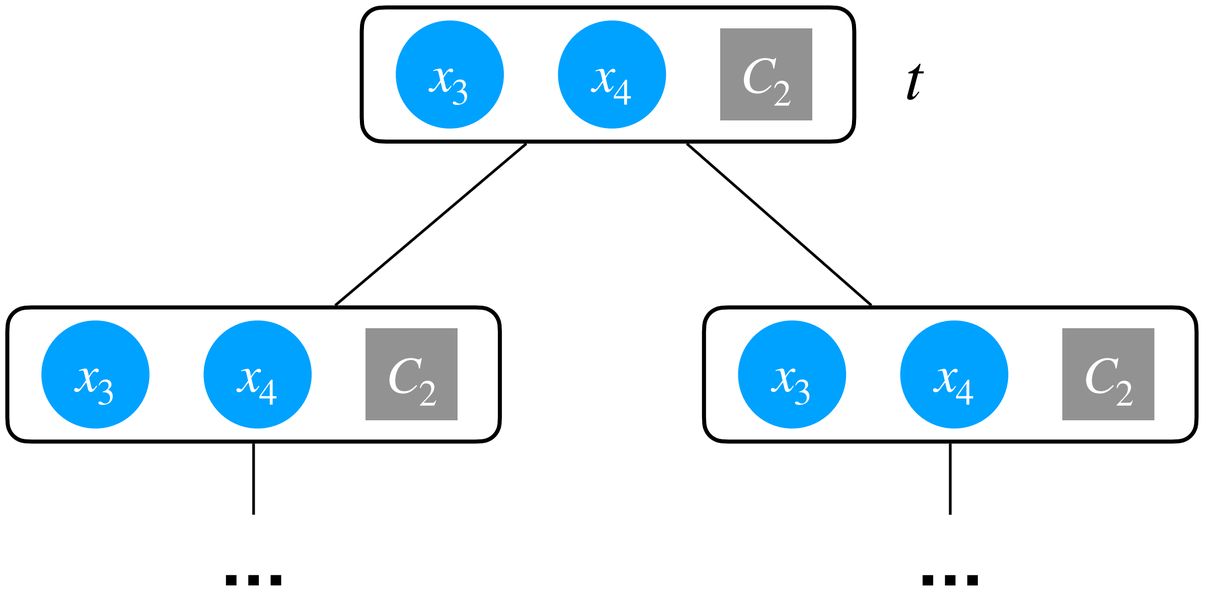

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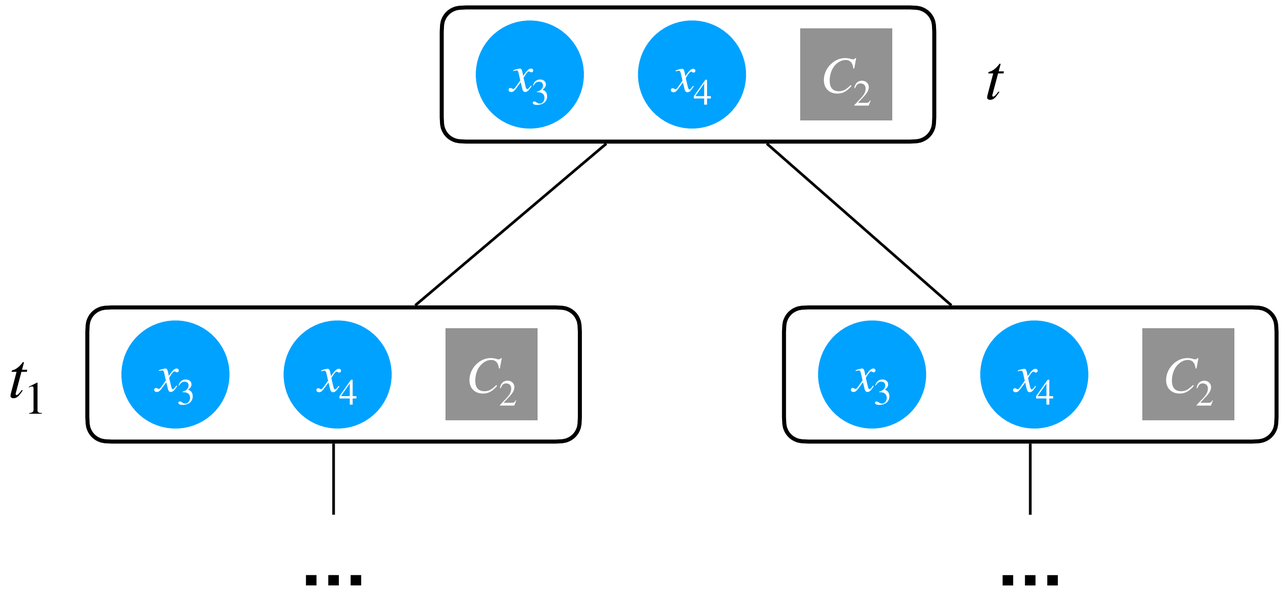

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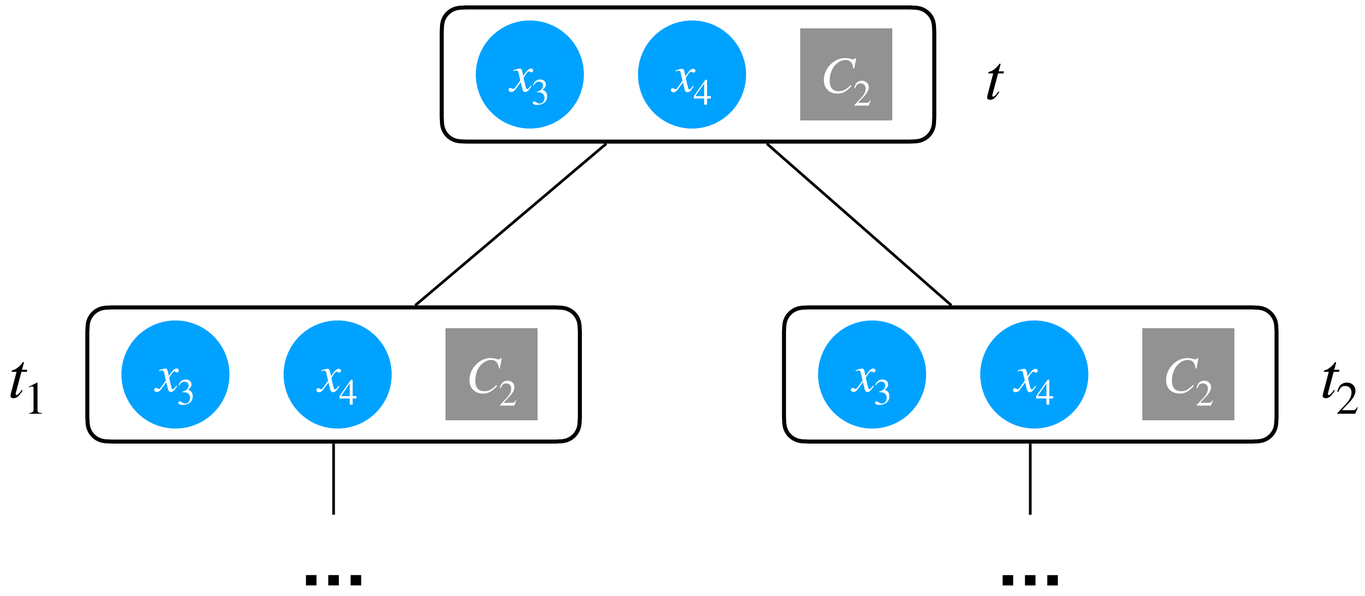

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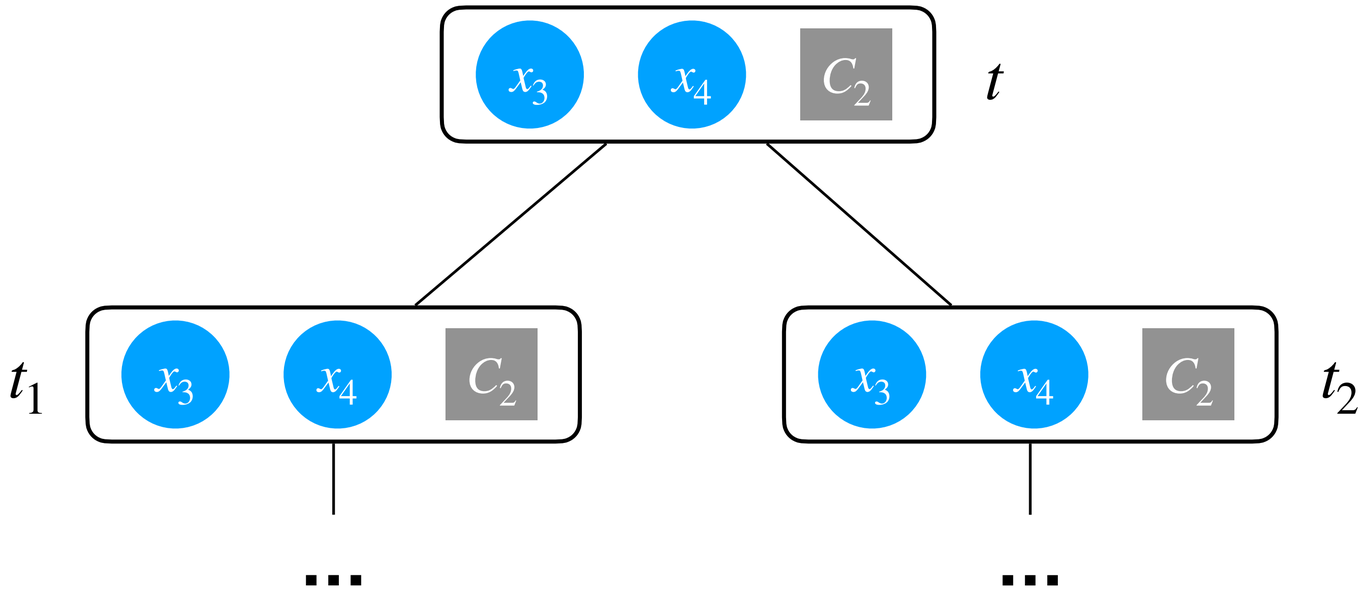

$$n(t, \alpha, A) = \sum_{\substack{A_1, A_2 \subseteq \chi_c(t) \\ A_1 \cap A_2 = A}} n(t_1, \alpha, A_1)\, n(t_2, \alpha, A_2)$$

Covering Product

$$n(t, \alpha, A) = \sum_{\substack{A_1, A_2 \subseteq \chi_c(t) \\ A_1 \cap A_2 = A}} n(t_1, \alpha, A_1)\, n(t_2, \alpha, A_2)$$

Covering Product

$f, g : 2^S \to \mathbb{Z}$

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$$(f *_c g)(Y) = \sum_{A \cup B = Y} f(A)\, g(B)$$

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Theorem (Björklund, Husfeldt, Kaski, Koivisto 2007)

All values of the covering product $f *_c g$ can be computed with $O(2^k k)$ arithmetic operations.

$$n(t, \alpha, A) = \sum_{\substack{A_1, A_2 \subseteq \chi_c(t) \\ A_1 \cap A_2 = A}} n(t_1, \alpha, A_1)\, n(t_2, \alpha, A_2)$$

Summary

#SAT with incidence treewidth k^*

$$4^{k^*} \cdot p(F)$$

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Fast Zeta- and Möbius Transform
Covering Product

Summary

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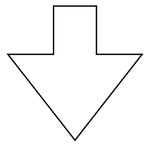

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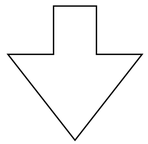

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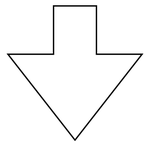

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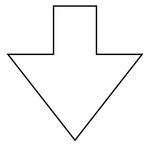

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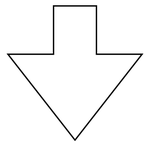

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