Simplified and Improved Separations Between Regular and General Resolution by Lifting

Marc Vinyals

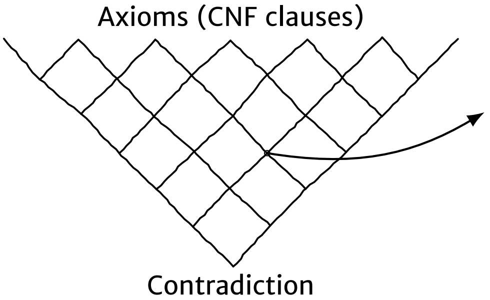
Technion
Haifa, Israel

joint work with Jan Elffers, Jan Johannsen, and Jakob Nordström

Background

Regular Resolution

'37 Resolution. [Blake] Res



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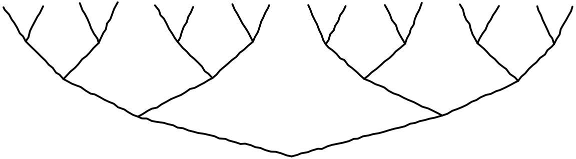

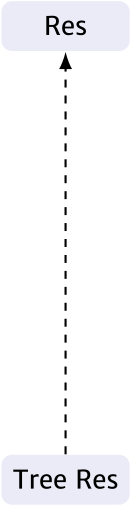

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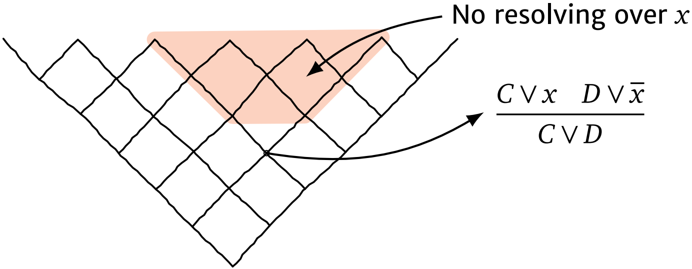

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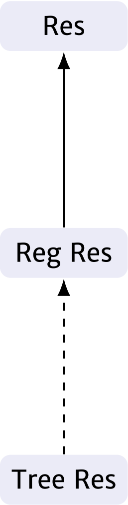

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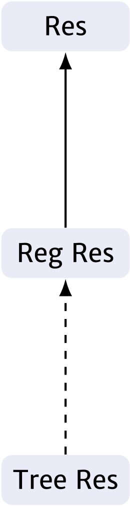

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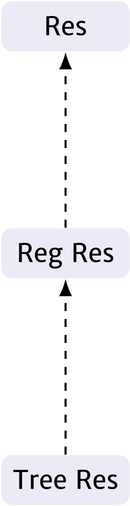
'87 Separation regular vs general (by a constant). [Huang, Yu]

'93 Separation regular vs general (superpolynomial). [Goerdt]

'02 Separation regular vs general (exponential). [AJPU]

'11 Best separation to date: $\exp(L/\log^7 L \log\log L)$. [Urquhart]



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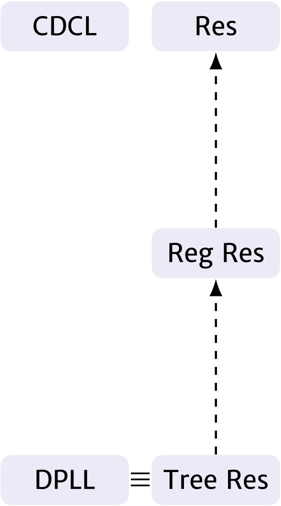
- Also: VSIDS, Restarts.



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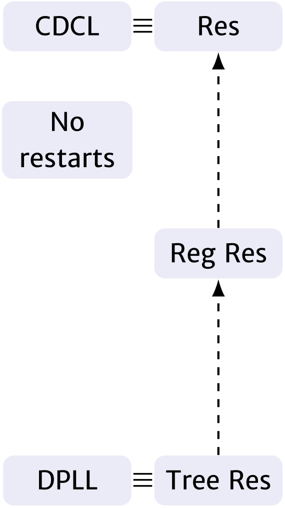

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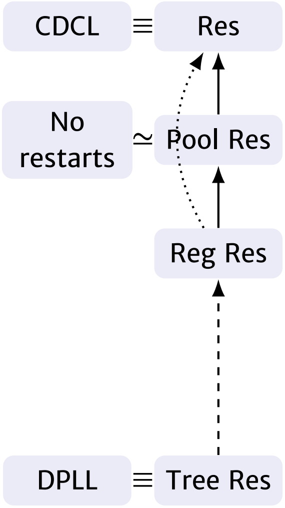
- Pool res $\geq$ Regular res $\Rightarrow$ Formulas that separate general and regular are good candidates to separate general and pool.



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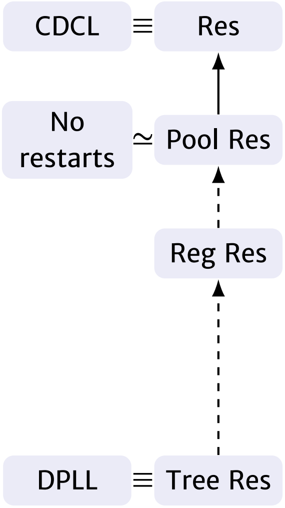
'14 All such formulas easy for pool resolution. [BBJ; BK]

- Also: formulas not good to run experiments with.
- Need new formulas!



Proving Resolution Lower Bounds

Largest clause in proof

Size–Width Relation

Resolution	F requires width $W \Rightarrow F$ requires length $\exp(W^2/n)$
Tree–like resolution	F requires width $W \Rightarrow F$ requires length $\exp(W)$
Regular resolution	??

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Lifting

Resolution	F requires width $W \Rightarrow T(F)$ requires length $\exp(W)$
Tree-like resolution	F requires depth $D \Rightarrow T(F)$ requires length $\exp(D)$
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Longest path in proof DAG

Results

Main Result (Informal)

Theorem

F requires large depth $\Rightarrow T(F)$ requires long regular proofs.

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- New family of "sparse stone formulas".
- Improved separation: $\exp(L/\log^3 L \log\log^5 L)$.
- Can use in experiments.

Lifting

Usual Lifting

- Replace each original variable x_i with a gadget $g_i(y_i^1, \ldots, y_i^k)$.
- e.g. $x_1 \lor \neg x_2 \quad \to \quad (y_1^1 \oplus y_1^2) \lor \neg(y_2^1 \oplus y_2^2)$.

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Lifting with Reusing

- Share variables among gadgets.

Lifting

Selector variables

Main variables

Lifting with Indexing

- Gadget $g_i(s_i^1, \ldots, s_i^m; r_i^1, \ldots, r_i^m)$: if $s_i^j = 1$, then $g_i(\cdots) = r_i^j$.
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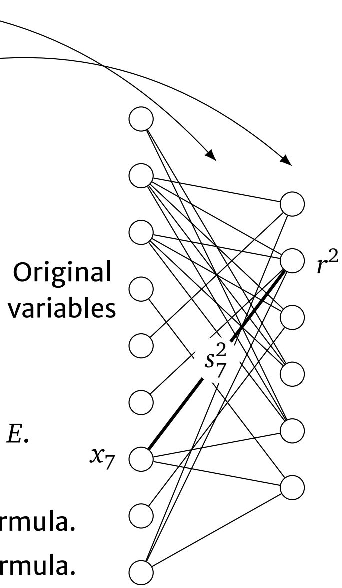
Lifting with Sparse Indexing and Reusing

- Fix a bipartite graph $G([n] \cup [m], E)$; variable s_i^j exists iff $(i,j) \in E$.
- G is n disjoint stars $\Rightarrow$ usual lifting.
- F is pebbling formula and G is complete graph $K_{n,m}$ $\Rightarrow$ stone formula.
- F is pebbling formula and G is random graph $\Rightarrow$ sparse stone formula.



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Theorem (Sparse)

If F requires depth D, then $\mathcal{L}_G(F)$ requires regular length $\sim \exp(D^3/n^2 \log^2 n)$.

G is a random graph of degree $d = \log(n/D)$.

Proof

Proof Overview (Dense)

Random restriction technique

1. Hit proof with random restriction ρ.
2. If proof of F is short, obtain proof of $F\restriction_\rho = F'$ with no wide clauses.
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- Need restriction to respect lifting: $\mathcal{L}(F)\restriction_\rho = F' = \mathcal{L}(F'')$. [AJPU '02]
- Need to tweak what "wide" means.
- Clause is "complex" if [AJPU '02]
  - talks about many main variables or
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Proof Overview (Dense)

Updated plan

1. Hit proof with lifting-respecting restriction ρ.
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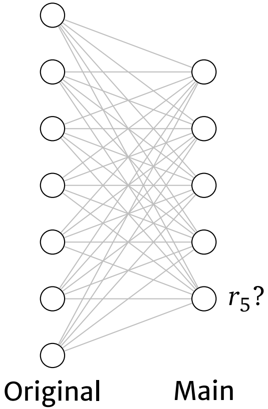

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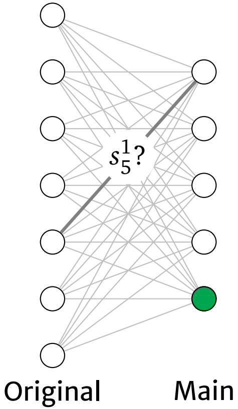

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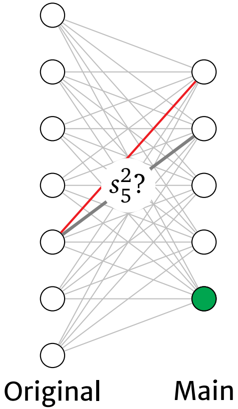

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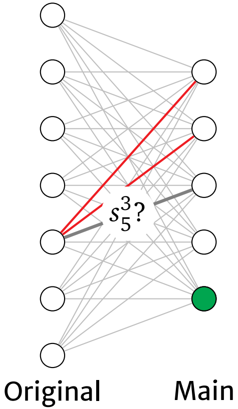

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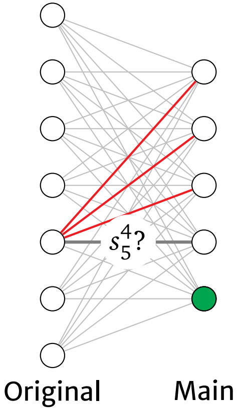

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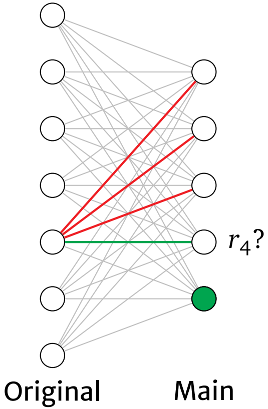

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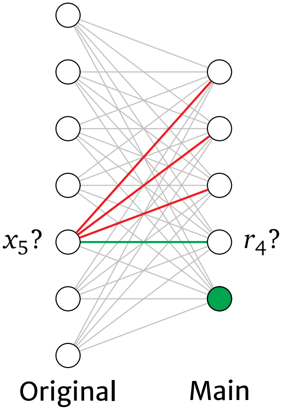

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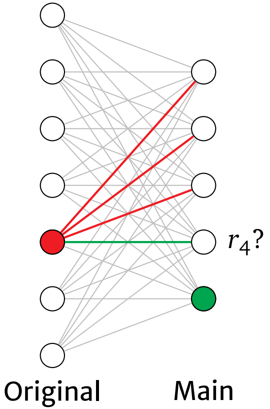

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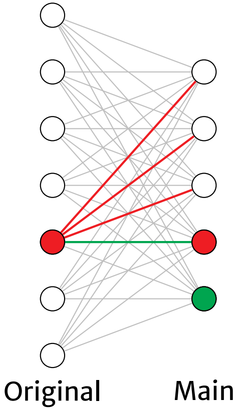
  - If query main variable: if matched, answer according to decision tree.
- If no complex clause, then a coloured main variable is never matched
  - Hence must query D main variables.
  - Hence (read once) must query D different main variables.
  - Contradiction, only have $m < D$ main variables.



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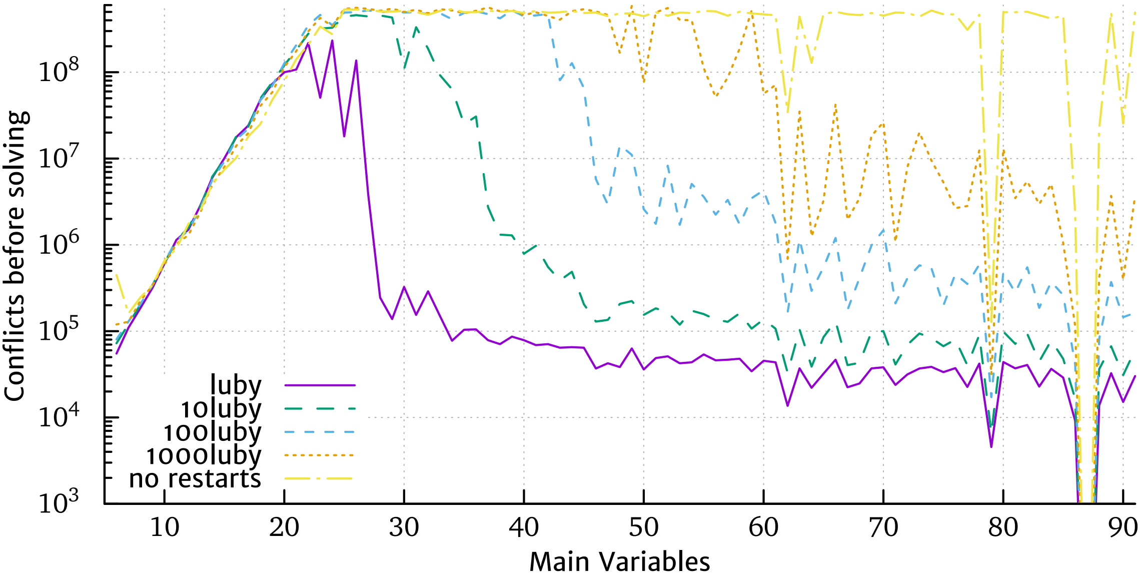
In practice. . .

- Few main variables ⇒ very hard.
- Many main variables ⇒ restarts crucial.

Sparse stone formula, base depth $D = 12$



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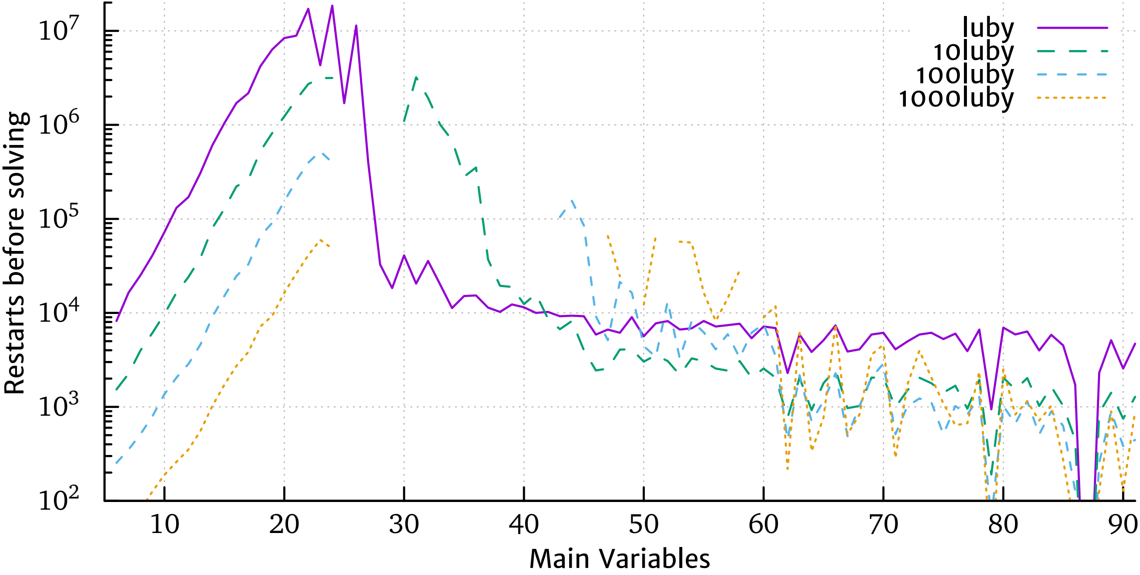

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